A SOLUTION TECHNIQUE FOR THE MINIMUM-TIME CONTROL PROBLEM OF AN R-THETA MANIPULATOR

by

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B.S.,Wichita State University, 1985

A MASTERS THESIS

submitted in partial fulfillment of

the requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1987

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ACKNOWLEDGEMENTS

I would like to thank the following people for their contributions to this thesis: Dr. Warren N. White, Jr., whose guidance and enthusiasm made this paper possible, Dr. J. Garth Thompson for his guidance in the optimal control theory concepts and for serving on my committee, Dr. Prakash Krishnaswami for his help with the minimization techniques and for serving on my committee, Dr. Ruth A. Dyer for serving on my committee, the Department of Mechanical Engineering, Kansas State University, and the Center of Excellence for Research in Computer-Controlled Automation, College of Engineering, Kansas State University and finally, my friends for their encouragement and support.

CHAPTER 1

INTRODUCTION

American industry, as a whole, and the automobile industry in particular, has been investing heavily in recent years to modernize manufacturing facilities, to lessen manufacturing costs, and to improve quality. Emphasis has been placed on teachable automation in an effort to ensure that new equipment will be flexible enough to accomodate future products with minimal additional investment. The primary teachable components of flexible manufacturing systems are robots, and the primary component of a robotic system that allows it to be teachable is computer control.

However, computer control of robots is not well refined and robots do not perform upto their physical capabilities. Higher performance robots would be valuable in manufacturing. For example, many applications of robots in factories can be justified economically only if implemented with a faster robot than state-of-the-art control permits. Other applications could be implemented with fewer faster robots, which would result in considerable investment savings. Another performance limitation, besides speed, is the maximum load bearing capacity of commercially available robots.

These performance limitations can be traced to the manner in which existing robots are controlled. State-of-the-art control schemes are based solely on stability requirements. The control law is designed to produce stability in axis position and then the fact of stability itself is used to induce motion by iteratively changing the position reference. This form of control gives rise to inaccuracy at high speeds and to position overshoots. Consequently robot designers have restricted the peak speed and acceleration of their products so that accuracy and overshoot can be limited to acceptable levels. This is a performance limitation due to the control laws and not due to the capabilities of the machine. Therefore, to control the robot for high performance, the true physical performance limitations must first be established. The limitations are then based on constraints and not merely on the control [1].

A real life example will illustrate this difference in performance levels. The standard PUMA arm equipment could allow base motor speeds of up to 144 rad/s, and currently, the limit set in the Puma controller is 89.9 rad/s [1]. A 60% improvement is possible if the full potential of the motor can be tapped.

The question of what is minimum time control can be answered with a commonly used analogy. If a person were travelling in a car and wished to get to the next intersection in the shortest possible time, what would he do? He would push the accelerator to the floor for a certain amount of time and then releasing the accelerator apply maximum braking (switch controls) for some other period of time to come to a stop. If the control switched too late, the car would slide into the

intersection. If it switched too soon, the car would stop short of the intersection. Thus, it is obvious that the switching time is critical in obtaining the desired final position and velocity. This is the basic idea in the minimum time optimal control of a robot manipulator or articulated mechanism. This type of control is also called bang-bang control [2], because the control is always on the control boundary, in one direction or the other.

The minimum time, optimal control problem for a robotic manipulator can be divided into two main classes, minimum time control along a specified path and minimum time control with no path constraints. Task oriented problems and obstacle avoidance planning fall into the former class. The second class can be subdivided into two categories, that of problems where the complete nonlinear minimum time system is considered and the true solution is sought, and of problems where approximations (usually linearizations) are made on the nonlinear system and the near-minimum time control is investigated. Extensive work has been done with problems belonging to the first class [3] - [7]. Problems dealing with the near-minimum time control have also been quite extensively investigated in recent years. In 1971 Kahn and Roth published a paper on the near-minimum time control of open loop articulated kinematic chains [8]. They developed a suboptimal feedback control by linearizing the equations of motion for a three degree of freedom manipulator. Approximations were made for the effects of gravity loads and angular velocities in the nonlinear dynamic equations. The suboptimal control was obtained by decoupling the system into three double integrators and deriving the equations for the switching curves

of the transformed system. The response time of the suboptimal control was compared to that of the optimal control which was obtained by an iterative technique. Wen and Desrochers [9] investigated two control strategies for suboptimal control, the method of averaging dynamics (AD) and the method of linear equivalence (LE). The first method is used when a time-fuel suboptimal solution is required. The latter uses exact linearization where the dynamic equations are written for a reduced system of decoupled double integrators. The LE method is found to be superior to the AD method in obtaining a smaller final time. However both methods need a very good model of the system since the nonlinear part of the system has to be evaluated repetitively. Sato, Shimojima and Kitamura [10] obtained switching times of the control variables for a two degree of freedom manipulator by approximating the velocity of a DC servomotor. They found that when the driving force was operating at saturation it was necessary to make additional approximations on the angular velocities. Kao, Sinha and Mahalanabis [11] developed an algorithm for the near-minimum time control of a three link robotic manipulator. They linearized the dynamic equations by expanding them in a Taylor series and neglecting the higher order terms. The poles of the linearized closed loop system were placed in the z-plane so as to permit minimum time response without violating the actuator torque constraints. This is a digital algorithm that can be implemented using microprocessors.

However very little work has been done in the area of the complete problem, the problem of determining the minimum time optimal control history for a system with no linearizations or approximations. Kahn and

Roth [8] obtained the minimum time optimal control by an iterative technique. Guesses were made on the unknown constants at the final time $\lambda(t_f)$, and the dynamic equations were integrated backwards in time to the initial time to give the states $x(t_o)$ and the constants $\lambda(t_o)$. If $x(t_o)$ is not sufficiently close to the specified initial state x_o then a new set of variables $\lambda(t_f)$ are chosen and the integration is repeated. The iteration is continued till $x(t_o)$ is sufficiently close to x_o. Then the constants $\lambda(t_o)$ and x_o are substituted into the dynamic equations and the optimal control is obtained by integrating the equations forward to the final time.

The purpose of this work is to develop an alternate method of solving the complete minimum time problem , to examine the deviation of the discrete time solution (finite element method) from the continuous case, and to explore the feasibility of a real-time minimum time controller.

In Chapter 2, the minimum time control problem is stated. The basic concepts of control theory, as well as some variational calculus principles, utilized in the problem formulation, are presented.

In Chapter 3, the mathematical model of the r-theta manipulator (a two degree of motion manipulator) is developed. The dynamic equations,derived using the Lagrangian formulation, are used in the control algorithm for the minimum time simulation of the manipulator.

In Chapter 4, the finite-element solution technique for the minimum time problem is developed.

In Chapter 5, the simulation results are presented. The finite element method is found to converge to the solution with reasonable initial guesses on unknown parameters. When this method is used in conjunction with a grid search method to start the algorithm, it converges quite rapidly to the true solution. The discrete time solution compares favorably with the continuous case, and as the grid density of the finite element mesh is increased the accuracy of the solution is improved.

Some of the limitations of the technique, as well as recommendations on areas for further investigations are also presented.

CHAPTER 2

THE MINIMUM TIME CONTROL PROBLEM

Optimal Control Theory

The objective of optimal control theory is to determine the control signals that will cause a process to satisfy the physical constraints and at the same time minimize (or maximize) some performance criterion.

In order to evaluate the performance of a system quantitatively, the designer has to select a performance index or cost function J. An optimal control is defined as one that minimizes (or maximizes) the performance index.

In the general case, it will be assumed that the performance of a system is evaluated by a measure of the form [12]

$$J = h(x(t_f), t_f) + \int_{t_o}^{t_f} g(x(t), u(t), t)\, dt \tag{2.1}$$

where t_o is the initial time, t_f is the final time, and h and g are scalar functions. The final time t_f may be specified or free depending on the problem statement. For the minimum time problem

$$J = \int_{t_o}^{t_f} 1\, dt \tag{2.2}$$

where t_f is unspecified.

Throughout this paper **bold face** characters will represent vectors. For example $\mathbf{x}(t)$ and $\mathbf{u}(t)$ are the state and control vectors.

The optimal control problem is to find an admissible control which causes the system described by the set of first order ordinary differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{2.3}$$

to follow an admissible trajectory $\mathbf{x}^*(t)$ that minimizes (or maximizes) the performance index J. The quantity $\mathbf{u}^*(t)$ which minimizes J is called the optimal control and $\mathbf{x}^*(t)$ an optimal trajectory. The elements of equation (2.3) are called state equations and involve the state variables $\mathbf{x}(t)$ and the controls $\mathbf{u}(t)$. A more formal definition of state variables is provided in Chapter 3.

A control history which satisfies the control constraints during the entire time interval $[t_o, t_f]$ and achieves the desired final state $\mathbf{x}(t_f)$ is called an admissible control. A state trajectory which satisfies the state variable constraints, both the differential equation constraints as well as the boundary constraints, during the entire time interval $[t_o, t_f]$ is called an admissible trajectory.

Variational Formulation

Variational calculus is a branch of mathematics that is very useful in solving optimization problems. The performance index J is a functional. A functional is a function of a function and/or functions.

For example, if

$$x_1 = f_1(q_1, q_2) \tag{2.4.1}$$

and

$$x_2 = f_2(q_1, q_2) \tag{2.4.2}$$

where q_1, q_2 are independant variables and f_1, f_2 are scalar functions

then the quantity

$$J = g(x_1, x_2) \tag{2.4.3}$$

is a functional where g is a scalar function.

Variation of a Functional

The variation of a functional plays the same role in determining extreme values (maximum or minimum) of a functional as does the differential in finding maxima or minima of functions. The differential, df, of a function, f, of variables $q_1, q_2, \ldots, q_n$ is given by the relation

$$df = \frac{\partial f}{\partial q_1} dq_1 + \frac{\partial f}{\partial q_2} dq_2 + \ldots + \frac{\partial f}{\partial q_n} dq_n. \tag{2.5}$$

Similarly, the variation, δJ, of a functional, J, of functions $x_1, x_2, \ldots, x_n$ is given by the relation

$$\delta J = \frac{\partial J}{\partial x_1} \delta x_1 + \frac{\partial J}{\partial x_2} \delta x_2 + \ldots + \frac{\partial J}{\partial x_n} \delta x_n. \tag{2.6}$$

Fundamental Theorem of the Calculus of Variations

The fundamental theorem states that the variation must be zero on an extremal (maximum or minimum) curve, provided there are no bounds imposed on the curves.

In other words

$$\delta J(x^*, \delta x) = 0 \tag{2.7}$$

for all <u>admissible δx</u>. By admissible δx, it is meant that $x + \delta x$ must be of some class Ω to which x belongs. For example, if Ω is the class of continuous functions, x and δx must both be continuous. In this case Ω comprises all the state histories which satisfy (2.3).

Constrained Minimization of Functionals

So far, functionals involving the state vector $x(t)$ have been discussed and it has been assumed that the components of $x(t)$ are independent. This is usually not the case in control problems where the state trajectory is determined by the control $u(t)$ and the state equations. Therefore it is necessary to consider functionals of n+m functions, $x(t)$ and $u(t)$, but only m of the functions are independent - the controls. The next step is to derive the necessary conditions for extremals of constrained systems. The Lagrangian multiplier method will be used.

The Lagrangian Multiplier Method for a System with Differential Equation Constraints

The objective is to find the necessary conditions for functions $x^*(t)$ and $u^*(t)$ to be extremals for a functional

$$J(x,u) = \int_{t_o}^{t_f} g(x(t),u(t),t)\, dt \tag{2.8}$$

where $x^*(t)$ is the state vector of order n and $u^*(t)$ is the control vector of order m. These vectors must also satisfy equation (2.3), the

differential equation constraints on the states. To include these constraints the augmented functional is formed . The augmented functional is defined as

$$J_a(\mathbf{x},\mathbf{u},\boldsymbol{\lambda}) \triangleq \int_{t_o}^{t_f}\{g(\mathbf{x}(t),\mathbf{u}(t),t)+ \boldsymbol{\lambda}^T(t)[\mathbf{a}(\mathbf{x}(t),\mathbf{u}(t),t)-\dot{\mathbf{x}}(t)]\}dt \quad (2.9)$$

where $\lambda_i(t)$ $i = 1,2,...,n$ are the Lagrangian multipliers whose values are to be determined. When the constraints are satisfied, the augmented functional, J_a, equals the functional, J, for any $\boldsymbol{\lambda}(t)$.

The quantity $g_a(\mathbf{x}(t),\dot{\mathbf{x}}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t)$can be defined as

$$g_a(\mathbf{x}(t),\dot{\mathbf{x}}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t) \triangleq g(\mathbf{x}(t),\mathbf{u}(t),t)$$

$$+ \boldsymbol{\lambda}^T(t)[\mathbf{a}(\mathbf{x}(t),\mathbf{u}(t),t)-\dot{\mathbf{x}}(t)] \quad (2.10)$$

so that

$$J_a(\mathbf{x}(t),\dot{\mathbf{x}}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t) = \int_{t_o}^{t_f} \{g_a(\mathbf{x}(t),\dot{\mathbf{x}}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t)\}dt. \quad (2.11)$$

The variation of the functional J_a, δJ_a ,after integrating by parts and simplifying is

$$\delta J_a = [\frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}(t_f),\dot{\mathbf{x}}(t_f),\mathbf{u}(t_f),\boldsymbol{\lambda}(t_f),t_f)]^T\delta\mathbf{x}_f+ [g_a(\mathbf{x}(t_f),\dot{\mathbf{x}}(t_f),$$

$$\mathbf{u}(t_f),\boldsymbol{\lambda}(t_f),t_f) - [\frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}(t_f),\dot{\mathbf{x}}(t_f),\mathbf{u}(t_f),\boldsymbol{\lambda}(t_f),t_f)]^T\dot{\mathbf{x}}(t_f)]\delta t_f$$

$$+ \int_{t_o}^{t_f}\{[[\frac{\partial g_a}{\partial \mathbf{x}}(\mathbf{x}(t),\dot{\mathbf{x}}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t)]^T$$

$$- \frac{d}{dt}[\frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}(t),\dot{\mathbf{x}}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t)]^T]\ \delta\mathbf{x}(t)$$

$$+ [\frac{\partial g}{\partial \mathbf{u}} a(\mathbf{x}(t),\dot{\mathbf{x}}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t)]^T \, \delta\mathbf{u}(t)$$

$$+ [\frac{\partial g}{\partial \boldsymbol{\lambda}} a(\mathbf{x}(t),\dot{\mathbf{x}}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t)]^T \, \delta\boldsymbol{\lambda}(t)\} \, dt. \tag{2.12}$$

The necessary conditions can be derived from the above equation by applying the fundamental theorem. However it is more convenient to use another functional, the Hamiltonian, which can be defined as [12]

$$H(\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t) \triangleq g(\mathbf{x}(t),\mathbf{u}(t),t) + \boldsymbol{\lambda}^T(t)[\mathbf{a}(\mathbf{x}(t),\mathbf{u}(t),t)] \tag{2.13}$$

where $\mathbf{a}(\mathbf{x}(t),\mathbf{u}(t),t)$ is the right hand side of equation (2.3).

For the minimum time problem,the Hamiltonian can be written as

$$H = 1 + \boldsymbol{\lambda}^T(t)[\mathbf{a}(\mathbf{x}(t),\mathbf{u}(t),t)]. \tag{2.14}$$

For an extremal curve the fundamental theorem gives us the condition

$$\delta J_a(\mathbf{x}^*(t),\mathbf{u}^*(t),\boldsymbol{\lambda}^*(t),t) = 0. \tag{2.15}$$

The superscript * signifies the extremal or optimal value.

The above equation gives us the necessary but not the sufficient conditions for optimal control which are

$$\dot{\mathbf{x}}(t) = \frac{\partial H}{\partial \boldsymbol{\lambda}} (\mathbf{x}^*(t),\mathbf{u}^*(t),\boldsymbol{\lambda}^*(t),t), \tag{2.16}$$

$$\dot{\boldsymbol{\lambda}}(t) = -\frac{\partial H}{\partial \mathbf{x}} (\mathbf{x}^*(t),\mathbf{u}^*(t),\boldsymbol{\lambda}^*(t),t), \tag{2.17}$$

$$\frac{\partial H}{\partial \mathbf{u}} (\mathbf{x}^*(t),\mathbf{u}^*(t),\boldsymbol{\lambda}^*(t),t) = 0 , \tag{2.18}$$

and

$$0 = [- \boldsymbol{\lambda}^*(t_f)]^T \delta \mathbf{x}_f + [H(\mathbf{x}^*(t_f),\mathbf{u}^*(t_f),\boldsymbol{\lambda}^*(t_f),t_f)]\delta t_f \, . \tag{2.19}$$

Equation (2.16) constitutes the n state equations. Equation (2.17) constitutes the n co-state equations. Equation (2.18) constitutes the m optimality conditions. Equation (2.19) constitutes the boundary condition equation. The conditions above are not sufficient to solve the minimum time problem because constraints on the controls are required to solve the problem. If the controls are unconstrained then the optimal control will be infinte torque or infinite force and the minimum time will be zero. The control constraints are defined in Chapter 3. For the minimum time problem the final time, t_f, is free to vary, but the final state $\mathbf{x}_f$ is fixed. Therefore

$$\delta \mathbf{x}_f = 0. \tag{2.20}$$

The boundary condition equation reduces to

$$H(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \boldsymbol{\lambda}^*(t_f), t_f) = 0. \tag{2.21}$$

This equation is also called the transversality equation.

So far it has been assumed that the admissible controls and states are not constrained by any boundaries, however, in realistic systems such constraints do commonly occur. Physically realizable controls generally have magnitude limitations. Actuators in robot joints have a maximum torque output beyond which they saturate. The generalization of the fundamental theorem to include the effects of the control boundary constraints leads to Pontryagin's minimum principle.

Pontryagin's minimum principle states that an <u>optimal control</u> must minimize the Hamiltonian, i.e.

$$H(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t) \leq H(\mathbf{x}^*(t), \mathbf{u}(t), \boldsymbol{\lambda}^*(t), t) \tag{2.22}$$

for any $t \varepsilon [t_o, t_f]$ and for all admissible controls.

The conditions for minimum time control, equations (2,16), (2,17), (2,18), (2,21) and (2.22) are utilized in the continuous time simulation and in the discrete time simulation of a robotic manipulator in Chapters 3 and 4, respectively.

CHAPTER 3

MATHEMATICAL MODEL

The Concept of State

The concept of state occupies a central position in modern control theory. It is a complete summary of the status of the system at a particular point in time. Knowledge of the state at some initial time t_o, plus knowledge of the system inputs after t_o, allows the determination of the state at some later time t_1. At any fixed time the state of a system can be described by the values of a set of variables x_i, $i = 1, 2, \ldots, n$ where n is the order of the system. These variables are called the state variables.

The Mathematical Model

An important part of any control problem is modelling the process. The objective is to obtain the simplest mathematical model that adequately predicts the response of the physical system to all anticipated inputs. The r-theta manipulator belongs to the class of systems that can be described by ordinary differential equations in state variable form. Thus if $x_1(t), x_2(t), \ldots, x_n(t)$ are the state variables of the process at time t and $u_1(t), u_2(t), \ldots, u_m(t)$ are the

control inputs to the process at time t, then the system may be described by n first order differential equations, such as

$$\begin{aligned}
\dot{x}_1(t) &= a_1(x_1(t),x_2(t),\ldots,x_n(t),u_1(t),u_2(t),\ldots u_m(t)) \\
\dot{x}_2(t) &= a_2(x_1(t),x_2(t),\ldots,x_n(t),u_1(t),u_2(t),\ldots u_m(t)) \\
\cdot \quad & \quad \cdot \\
\cdot \quad & \quad \cdot \\
\dot{x}_n(t) &= a_n(x_1(t),x_2(t),\ldots,x_n(t),u_1(t),u_2(t),\ldots u_m(t))
\end{aligned} \qquad . \ (3.1)$$

The state vector $\mathbf{x}(t)$ of the system is defined as

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ x_n^{\cdot}(t) \end{bmatrix} \qquad (3.2)$$

and the control vector is defined as

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ u_m^{\cdot}(t) \end{bmatrix} \quad . \qquad (3.4)$$

The state equations in vector form are

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t),\mathbf{u}(t),t). \qquad (3.4)$$

Kinematic Model

The two degrees-of-freedom robotic manipulator on which the minimum time control is performed is called an r-theta manipulator. A schematic of the manipulator is shown in Figure 3.1. The kinematic model is illustrated in Figure 3.2.

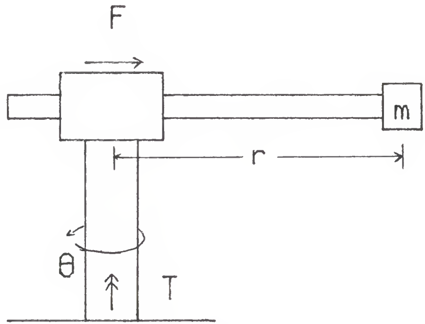

FIGURE 3.1 THE R-THETA MANIPULATOR

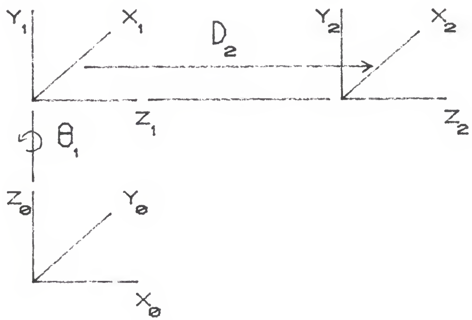

FIGURE 3.2 KINEMATIC MODEL

The r-theta manipulator has two joints. Joint 1 is revolute and joint 2 is prismatic. The joint variables are θ and r (or D_2 in the Denavit-Hartenberg representation [13] used in Figure 3.2), respectively. The torque on joint 1 is T and the force on joint 2 is F. The plane of motion of the manipulator is parallel to the ground and hence the gravitational force does not enter into the dynamic equations.

Dynamic Model

The equations of motion are nonlinear and coupled. In order to simplify the equations, and hence the simulation, the following assumptions are made.

1. The mass of the payload is much greater than the mass of the links and actuators.
2. The payload m is treated as a point mass.

The first assumption allows the r-theta links to be treated as massless kinematic linkages. The second assumption simplifies the inertia terms.

The dynamic equations are derived from Lagrange's equation of motion. If not all the forces acting on the system are derivable from a potential, then Lagrange's equations can be written in the form [14]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \left(\frac{\partial L}{\partial q_j}\right) = Q_j \tag{3.5}$$

where L is the Lagrangian, q_j represents the generalized coordinates, and Q_j represents the forces not arising from a potential.

If KE is the kinetic energy and PE the potential energy then the Lagrangian is defined as [14]

$$L = KE - PE \ . \tag{3.6}$$

Summing up the kinetic energy of the manipulator gives

$$KE = \frac{1}{2}\, m(\dot{r})^2 + \frac{1}{2}\, m(r\dot{\theta})^2 \tag{3.7}$$

where m is the mass of payload, θ is the joint 1 variable, $\dot{\theta}$ is the time derivative of θ , namely $\frac{d}{dt}(\theta)$, r is the joint 2 variable, and $\dot{r}$ is the time derivative of r , namely $\frac{d}{dt}(r)$. The potential energy of the manipulator is

$$PE = 0 \ . \tag{3.8}$$

The quantities r and θ are explicit functions of time, t. Throughout this chapter the independant variable t is omitted from the notation of the explicit functions of t for brevity. Also, throughout this chapter the superscript $\dot{}$ will indicate the first differential with respect to time $\frac{d}{dt}$ and the superscript $\ddot{}$ will indicate the second differential with respect to time $\frac{d^2}{dt^2}$.

The forces at the joints are

$$Q_1 = T \tag{3.9}$$

and

$$Q_2 = F \tag{3.10}$$

where T is the torque at joint 1 and F is the force at joint 2. Substituting equations (3.7) and (3.8) in equation (3.6) gives the Lagrangian

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) \ . \tag{3.11}$$

For the r-theta manipulator, the independent generalized coordinates are r and θ. The Lagrangian equations for the r-theta manipulator are of the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \left(\frac{\partial L}{\partial r}\right) = F \tag{3.12}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) = T \ . \tag{3.13}$$

From equation (3.11) the expressions for the various derivative terms of equations (3.12) and (3.13) are

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 \ , \tag{3.14}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{d}{dt}(m\dot{r}) = m\ddot{r} \ , \tag{3.15}$$

$$\frac{\partial L}{\partial \theta} = 0 \ , \tag{3.16}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}(mr^2\dot{\theta}) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} \ . \tag{3.17}$$

Substituting equations (3.14) and (3.15) into equation (3.12) gives

$$m\ddot{r} - mr\dot{\theta}^2 = F \ . \tag{3.18}$$

Substituting equations (3.16) and (3.17) into equation (3.13) gives

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = T \,. \tag{3.19}$$

The controls are defined as

$$u_1 = \frac{F}{Fmax} \tag{3.20}$$

and

$$u_2 = \frac{T}{Tmax} \tag{3.21}$$

where u_1 is the control at joint 2, u_2 is the control at joint 1, T is the actual torque applied at joint 1, Tmax is the maximum torque that can be applied at joint 1, F is the actual force applied at joint 2, and Fmax is the maximum force that can be applied at joint 2. The controls u_1 and u_2 are explicit functions of t.

Substituting equations (3.21) and (3.20) into equations (3.19) and (3.18), respectively, gives

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - Tmax\ u_2 = 0 \tag{3.22}$$

and

$$m\ddot{r} - mr\dot{\theta}^2 - Fmax\ u_1 = 0. \tag{3.23}$$

The state variables x_1, x_2, x_3, and x_4 are now introduced. They are defined as

$$x_1 = r, \tag{3.24.1}$$

$$x_2 = \dot{r}, \tag{3.24.2}$$

$$x_3 = \theta, \tag{3.24.3}$$

and

$$x_4 = \dot{\theta} \,. \tag{3.24.4}$$

From equations (3.22),(3.23) and (3.24) the state equations can be written as

$$\dot{x}_1 = x_2, \tag{3.25.1}$$

$$\dot{x}_2 = x_1 x_4^2 + \frac{Fmax}{m} u_1 \quad , \tag{3.25.2}$$

$$\dot{x}_3 = x_4, \tag{3.25.3}$$

and

$$\dot{x}_4 = -\frac{2x_2 x_4}{x_1} + \frac{Tmax}{mx_1^2} u_2 \tag{3.25.4}$$

where x_1, x_2, x_3, x_4 are explicit functions of t.

In this paper the performance index J_a is formulated in 2 ways which are :

1. The **First Order Formulation** where the performance index J is augmented with first order ordinary differential equation state constraints (refer to equation (2.9)). This formulation is used in the continuous time simulation (numerically integrated using the Runge-Kutta method).
2. The **Second Order Formulation** where the performance index J is augmented with second order ordinary differential equation state constraints. This formulation is used in the discrete time simulation (using Finite Element methods).

<u>The First Order Formulation</u>

From equations (2.9) and (3.25) the augmented performance index J_a can be written as

$$J_a(\mathbf{x},\dot{\mathbf{x}},\mathbf{u},\boldsymbol{\lambda}) = \int_{t_o}^{t_f} \{1 + \lambda_1^1[x_2 - \dot{x}_1] + \lambda_2^1[x_1 x_4^2 + \frac{Fmax}{m} u_1 - \dot{x}_2]$$

$$+ \lambda_3^1[x_4 - \dot{x}_3] + \lambda_4^1[-\frac{2x_2x_4}{x_1} + \frac{Tmax}{mx_1^2} u_2 - \dot{x}_4]\}dt \qquad (3.26)$$

where the superscript 1 on the λ's signifies first order formulation and where λ_1^1, λ_2^1, λ_3^1, λ_4^1 are the Lagrangian multipliers which are explicit functions of t.

From equations (2.14) and (3.25) the Hamiltonian for the r-theta manipulator can be defined as

$$H(\mathbf{x},\mathbf{u},\boldsymbol{\lambda}) = 1 + \lambda_1^1 x_2 + \lambda_2^1[x_1 x_4^2 + \frac{Fmax}{m} u_1] + \lambda_3^1 x_4$$

$$+ \lambda_4^1[-\frac{2x_2x_4}{x_1} + \frac{Tmax}{mx_1^2} u_2] \quad . \qquad (3.27)$$

Substituting equation (3.27) in equation (2.17) gives the co-state equations for the r-theta manipulator as

$$\dot{\lambda}_1^1 = - \lambda_2^1 x_4^2 - \lambda_4^1[\frac{2x_2x_4}{x_1^2} - \frac{2Tmax}{mx_1^3} u_2], \qquad (3.28)$$

$$\dot{\lambda}_2^1 = - \lambda_1^1 - \lambda_4^1[-\frac{2x_4}{x_1}] , \qquad (3.29)$$

$$\dot{\lambda}_3^1 = 0 , \qquad (3.30)$$

and
$$\dot{\lambda}^1_4 = -\lambda^1_2[2x_1x_4] - \lambda^1_3 - \lambda^1_4\left[\frac{-2x_2}{x_1}\right] . \tag{3.31}$$

Substituting equation (3.27) into equation (2.21) gives the transversality condition for the r-theta manipulator as

$$0 = 1 + \lambda^1_1 x_2 + \lambda^1_2\left[x_1x_4^2 + \frac{Fmax}{m}u_1\right] + \lambda^1_3 x_4 + \lambda^1_4\left[\frac{(-2x_2x_4)}{x_1} + \frac{Tmax}{mx_1^2}u_2\right] \tag{3.32}$$

at $t = t_f$.

For the simulation example it is assumed that the manipulator starts from rest and come to a stop at the final state. Therefore, the state variables

$$x_2(t_f) = 0 \tag{3.33}$$

and
$$x_4(t_f) = 0 . \tag{3.34}$$

Substituting equations (3.33) and (3.34) into equation (3.32) gives

$$1 + \lambda^1_2\left[\frac{Fmax}{m}u_1\right] + \lambda^1_4\left[\frac{Tmax}{mx_1^2}u_2\right] = 0. \tag{3.35}$$

The Second Order Formulation

The augmented functional J_a is defined in terms of the second order differential equation constraints (3.22) and (3.23) as

$$J_a(x,\dot{x},u,\lambda,t) = \int_{t_o}^{t_f} \{1+ \lambda_1[F\max\ u_1 - m\ddot{r} + mr\dot{\theta}^2]$$

$$+ \lambda_2[T\max\ u_2 - \frac{d}{dt}(mr^2\dot{\theta})]\}\ dt \qquad (3.36)$$

where the λ's with no superscripts signify the second order formulation.

Taking the variation J_a and applying the fundamental theorem gives the two multiplier equations and the transversality equation (refer to Appendix I). The multiplier equations are the second order differential equations in the Lagrangian multipliers. They are given by the following relations

$$- m\ \ddot{\lambda}_1 + m\lambda_1\dot{\theta}^2 + \dot{\lambda}_2 2mr\dot{\theta} = 0 \qquad (3.37)$$

and
$$2m[\lambda_1\{\dot{r}\dot{\theta} + r\ \ddot{\theta}\ \} + \dot{\lambda}_1 r\dot{\theta}] + \ddot{\lambda}_2 mr^2 + \dot{\lambda}_2 2mr\dot{r} = 0\ . \qquad (3.38)$$

When the velocities at the final state are zero the transversality equation is

$$1 + \lambda_1[F\max\ u_1\ ] + \lambda_2[T\max\ u_2\ ] = 0 \qquad (3.39)$$

which is eqvivalent to the first order formulation (equation (3.35)).

The Optimality Conditions for a Problem with Inequality Constraints

Figure 3.3 [2] provides one-dimensional illustrations of two possible types of minima with inequality constraints. It is required to minimize a functional I(u) subject to the inequality constraints

$$f(u) \leq 0 \qquad (3.40)$$

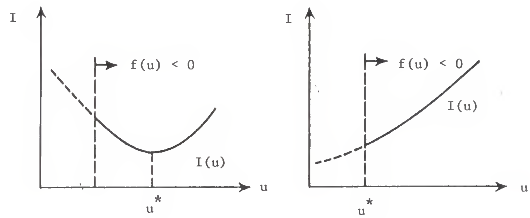

Figure 3.3. One-dimensional illustration of two possible types of minima with inequality constraints.

where in general $\mathbf{f}$ and $\mathbf{u}$ are vectors of different dimensions. The constraints can be appended to the functional $I(\mathbf{u})$ to give the augmented functional I_a

$$I_a(\mathbf{u},\boldsymbol{\mu}) = I(\mathbf{u}) + \boldsymbol{\mu}\mathbf{f}(\mathbf{u}) \tag{3.41}$$

where $\boldsymbol{\mu}$ is the vector of the inequality constraint multipliers. When these constraints are satisfied

$$I_a(\mathbf{u},\boldsymbol{\mu}) = I(\mathbf{u}) \ . \tag{3.42}$$

There are two cases for the optimal value of $\mathbf{u}$, $\mathbf{u}^*$, which are

$$\mathbf{f}(\mathbf{u}^*) < 0 \tag{3.43}$$

and

$$\mathbf{f}(\mathbf{u}^*) = 0 \ . \tag{3.44}$$

In the former case $\boldsymbol{\mu} = 0$ so that equation (3.42) is satisfied. In the latter case consider small perturbations about $\mathbf{u}^*$. If $I(\mathbf{u}^*)$ is a minimum, then

$$\delta I = \frac{\partial I}{\partial \mathbf{u}} \delta\mathbf{u} \geq 0 \tag{3.45}$$

for all admissible values of $\delta\mathbf{u}$, which must also satisfy

$$\delta \mathbf{f} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \delta\mathbf{u} \leq 0 \ . \tag{3.46}$$

For equations (3.45) and (3.46) to be true they must be of opposite sign which indicates

$$\operatorname{sgn}\left(\frac{\partial I}{\partial \mathbf{u}}\right) = -\operatorname{sgn}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right) \tag{3.47}$$

or

$$\frac{\partial I}{\partial \mathbf{u}} = 0 \tag{3.48}$$

where the signum function, sgn, is the sign of the argument and

where $\left(\frac{\partial I}{\partial \mathbf{u}}\right)$ is the negative linear combination of $\frac{\partial f_1}{\partial \mathbf{u}}, \frac{\partial f_2}{\partial \mathbf{u}}, \ldots, \frac{\partial f_m}{\partial \mathbf{u}}$

where m is the number of constraints. Equations (3.47) and (3.48) can be combined to give

$$\frac{\partial I}{\partial \mathbf{u}} + \mu \frac{\partial \mathbf{f}}{\partial \mathbf{u}} = 0 \tag{3.49}$$

for

$$\mu \geq 0. \tag{3.50}$$

Therefore the necessary conditions for minimizing $I(\mathbf{u})$ are

$$\frac{\partial I_a}{\partial \mathbf{u}} = 0 \tag{3.51}$$

and

$$\mathbf{f}(\mathbf{u}) \leq 0 \tag{3.52}$$

subject to the conditions

$$\mu \geq 0 \tag{3.53.1}$$

for

$$\mathbf{f}(\mathbf{u}) = 0 \tag{3.53.2}$$

and

$$\mu = 0 \tag{3.53.3}$$

for

$$\mathbf{f}(\mathbf{u}) < 0. \tag{3.53.4}$$

For minimum time problems bang-bang control is used, that is

$$\mathbf{u} = \pm \mathbf{u}_{max} \tag{3.54}$$

where $\mathbf{u}$ is the control vector and $\mathbf{u}_{max}$ is the maximum control magnitude. For the r-theta manipulator

$$\mathbf{u}_{max} = 1 . \tag{3.55}$$

From equations (3.52),(3.54) and (3.55) the inequality control constraints for the r-theta manipulator are

$$f_1(u_1) = u_1 - 1 , \tag{3.56.1}$$

$$f_2(u_1) = -u_1 - 1 \ , \tag{3.56.2}$$

$$f_3(u_2) = u_2 - 1 \ , \tag{3.56.3}$$

and

$$f_4(u_2) = -u_2 - 1 \ . \tag{3.56.4}$$

The functional I for the first order formulation is equal to H in equation (3.27)

$$I(x,u,\lambda) = 1 + \lambda_1^1 x_2 + \lambda_2^1 [x_1 x_4^2 + \frac{Fmax}{m} u_1 \ ] + \lambda_3^1 x_4$$

$$+ \lambda_4^1 [-\frac{2x_2x_4}{x_1} + \frac{Tmax}{mx_1^2} u_2 \ ] \ . \tag{3.57}$$

From equations (3.41) and (3.57) I_a is

$$I_a(x,u,\lambda,\mu,t) = I + \mu_1(u_1-1) + \mu_2(-u_1-1)$$

$$+ \mu_3(u_2-1) + \mu_4(-u_2-1) \ . \tag{3.58}$$

Substituting equation (3.58) into equation (3.51) gives

$$\frac{\partial I_a}{\partial u_1} = \lambda_2^1 \frac{Fmax}{m} + \mu_1 - \mu_2 = 0 \tag{3.59}$$

and

$$\frac{\partial I_a}{\partial u_2} = \lambda_4^1 \frac{Tmax}{mx_1^2} + \mu_3 - \mu_4 = 0 \tag{3.60}$$

subject to the conditions (3.52), and (3.53). From equation (3.59)

if

$$u_1 = 1 \tag{3.61.1}$$

then

$$\mu_1 \geq 0 \tag{3.61.2}$$

and $$\mu_2 = 0 \; . \tag{3.61.3}$$

Therefore $$\lambda_2^1 \leq 0 \tag{3.61.4}$$

in order for equation (3.59) to be true. If

$$u_1 = -1 \tag{3.62.1}$$

then $$\mu_1 = 0 \tag{3.62.2}$$

and $$\mu_2 \geq 0 \; . \tag{3.62.3}$$

Therefore $$\lambda_2^1 \geq 0 \tag{3.62.4}$$

in order for equation (3.59) to be true. From equations (3.61) and (3.62) the control u_1 can be defined as

$$u_1 = -\frac{|\lambda_2^1|}{\lambda_2^1} = -\mathrm{sgn}(\lambda_2^1) \tag{3.63}$$

for $$\lambda_2^1 \neq 0 \; . \tag{3.64}$$

The signum function, sgn, takes on a value which is equal to the sign of the argument. Similarily from equation (3.60) the control u_2 is defined as

$$u_2 = -\frac{|\lambda_4^1|}{\lambda_4^1} = -\mathrm{sgn}(\lambda_4^1) \tag{3.65}$$

for $$\lambda_4^1 \neq 0 \,. \tag{3.66}$$

When λ_2^1 or λ_4^1 is zero the respective control can move away from the constraint boundary. However for the r-theta manipulator λ_2^1 and λ_4^1 are at zero for only an instant and therefore the effect of control at those instances is not significant.

For the second order formulation the control u_1 is defined as

$$u_1 = -\frac{|\lambda_1|}{\lambda_1} = -\mathrm{sgn}(\lambda_1) \tag{3.67}$$

for $$\lambda_1 \neq 0 \tag{3.68}$$

and the control u_2 is defined as

$$u_2 = -\frac{|\lambda_2|}{\lambda_2} = -\mathrm{sgn}(\lambda_2) \tag{3.69}$$

for $$\lambda_2 \neq 0 \,. \tag{3.70}$$

These optimality conditions are used in the continuous time and discrete time simulations.

CHAPTER 4

THE FINITE ELEMENT METHOD

Introduction

In Chapter 3 the mathematical model of the manipulator was formulated. The state equations, the multiplier equations, the transversality equation, and the optimality conditions were derived for the r-theta manipulator by both the first and the second order formulations. The first order formulation is used in the continuous time simulation where the set of eight first order differential equations are numerically integrated using the Runge-Kutta method. The guesses on the unknown parameters (the Lagrangian multipliers at the initial time t_o and the final time t_f) are iterated upon using conventional minimization techniques like the conjugate gradient method [15], the Quasi-Newton method [16], and the Newton-Raphson method [17]. However none of the methods converge to the solution if the initial guesses on the unknown parameters are not sufficiently close to the optimal values.

A more robust method of solving the minimum time problem, a two point boundary value problem (TPBVP), is needed. Both the optimal control problem and the finite element method can be developed from variational principles. The finite element method has been successfully applied to a wide range of nonlinear problems as well as to the TPBVP.

Therefore, this method was applied to the solution of the minimum time problem with the objective of investigating the advantages and drawbacks of a discrete time simulation as compared to the continuous case. A flow chart of the finite element program is included in Appendix II.

Finite Element Analysis

The finite element method is a numerical analysis technique for obtaining approximate solutions to a wide range of engineering problems. In nonlinear problems, as in this particular case, closed form solutions are not available, so it is necessary to obtain approximate numerical solutions. Two of the more commonly used methods are the finite difference and the finite element methods [18]. For some problems, especially problems with irregular geometry or unusual boundary conditions, the finite element method is superior to the finite difference method.

The finite element method takes a continuum problem and discretizes the solution region into a finite number of elements. By expressing the unknown solution within each element in terms of assumed approximating functions called interpolation functions, the infinite number of unknowns in terms of the Taylor series expansion is reduced to a finite number.

One of the advantages of this method is the ability to formulate the properties of the individual elements, before putting them together to represent the entire problem. In effect, a complex problem is reduced to considering a series of greatly simplified problems. Another advantage is the variety of ways in which the problem can be formulated.

These include the direct approach (from physical laws), the variational approach, the weighted residual approach, and the energy balance approach [18]. In this paper the variational approach was used in determining the element properties.

There are 5 basic steps to the finite element method. These are:

1. Discretization of the continuum.
2. Selection of the interpolation function.
3. Determination of the element properties.
4. Assembly of element properties to obtain system equations.
5. Application of boundary conditions and solution of system unknowns.

Discretization of the Continuum

The first step is the discretization of the solution region into elements. The range of the independent variable, time t, from the initial state to the final state is discretized into elements of uniform length Δt. These elements are connected to adjoining elements by sharing common nodes. The element length, Δt, varies with change in grid density or the final time. The element unknowns are the position coordinates (r,θ) and the Lagrangian multipliers (λ_1, λ_2) at each node and the length of element (Δt). The performance index J_a will be expressed in terms of the approximations so that the continuous time

problem of minimizing J_a over the time interval $[t_o, t_f]$ is reduced to one of minimizing J_a for each element in the time domain.

Discretizing equation (3.35) gives

$$J_a = \sum_{i=1}^{n} \int_{(i-1)\Delta t}^{i\Delta t} \{1 + \lambda_1 [F\max u_1 + mr\dot{\theta}^2 - m\ddot{r}]$$

$$+ \lambda_2 [T\max u_2 - \frac{d}{dt}(mr^2\dot{\theta})]\} \, dt \qquad (4.1)$$

where n is the number of elements and Δt is the length of each element and $\sum_{i=1}^{n}$ is the summation over the elements 1 to n.

Selection of the Interpolation Function

The next step is to assign nodes to each element (points in time) and choose the interpolation function to represent the variation of the unknown variables over the elements. The state variables r, $\dot{r}$, θ, $\dot{\theta}$ and the multipliers and their time derivatives λ_1 λ_2, $\dot{\lambda}_1$ $\dot{\lambda}_2$ will be represented by linear interpolation functions of the form

$$x(t) = N_1(t)x_1 + N_2(t)x_2 \, , \qquad (4.2)$$

and

$$\dot{x}(t) = \frac{x_2 - x_1}{\Delta t} \qquad (4.3)$$

where x_1, x_2 are the values of the given unknowns $x(t)$ at nodes 1 and 2 of each element and the natural coordinates N_1, N_2 vary as shown in Figure 4.1

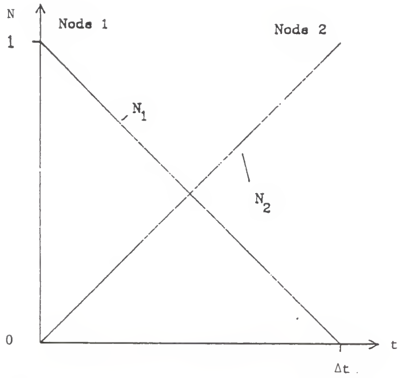

Figure 4.1: Natural Coordinates for First Order, One Dimensional Finite Element

The natural coordinates are defined by the relations

$$N_1 = 1 - \frac{t}{\Delta t} \tag{4.4}$$

and

$$N_2 = \frac{t}{\Delta t} \quad . \tag{4.5}$$

Therefore,

$$\int_0^{\Delta t} \{N_1\}\ dt = \int_0^{\Delta t} \{N_2\}\ dt = \frac{\Delta t}{2} \ , \tag{4.6}$$

$$\int_0^{\Delta t} \{N_1 N_2\}\ dt = \frac{\Delta t}{6} \ , \tag{4.7}$$

and

$$\int_0^{\Delta t} \{N_1^2\}\ dt = \int_0^{\Delta t} \{N_2^2\}\ dt = \frac{\Delta t}{3} \ . \tag{4.8}$$

The relations (4.6), (4,7) and (4.8) will be extensively used in the development of the element equations.

From equations (4.2) and (4.3) the unknown variables are defined as

$$r(t) = N_1 r_1 + N_2 r_2 \ , \tag{4.9.1}$$

$$\theta(t) = N_1 \theta_1 + N_2 \theta_2 \ , \tag{4.9.2}$$

$$\lambda_1(t) = N_1 \lambda_{11} + N_2 \lambda_{12}, \tag{4.9.3}$$

$$\lambda_2(t) = N_1 \lambda_{21} + N_2 \lambda_{22}, \tag{4.9.4}$$

$$\dot{r}(t) = \frac{r_2 - r_1}{\Delta t} \ , \tag{4.9.5}$$

$$\dot{\theta}(t) = \frac{\theta_2 - \theta_1}{\Delta t} \ , \tag{4.9.6}$$

$$\dot{\lambda}_1(t) = \frac{\lambda_{12} - \lambda_{11}}{\Delta t} \ , \tag{4.9.7}$$

and $$\dot{\lambda}_2(t) = \frac{\lambda_{22}-\lambda_{21}}{\Delta t} \qquad (4.9.8)$$

where λ_{11} and λ_{12} are the values of $\lambda_1(t)$ at nodes 1 and 2, respectively, of each element while λ_{21} and λ_{22} are the values of $\lambda_2(t)$ at the respective node designated by the second subscript.

Determination of the Element Properties

The variational approach will be used in the formulation of the element properties. The equations for the minimization of the performance index J_a over a single element will be derived in this section. In the next section the elements will be assembled to give the equations for the minimization of J_a over the entire time period $[t_o,t_f]$.

J_a in equation (4.1) is simplified by integrating by parts the second derivative terms in the equation. This gives rise to two boundary terms. For interior elements the boundary terms cancell with those from the adjoining elements. For exterior elements these terms go to zero because $\dot{r}(0)$, $\dot{\theta}(0)$, $\dot{r}(t_f)$ and $\dot{\theta}(t_f)$ are specified as zero in the boundary conditions. Considering equation (4.1) for a single element and substituting equations (4.6) - (4.9) into (4.1) gives

$$J_a = \Delta t + \frac{m}{6\Delta t}[\lambda_{11}\ \lambda_{12}] \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} [\theta_1\ \theta_2] \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

$$+ \frac{m}{6\Delta t}[r_1\ r_2] \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} [\lambda_{21}\ \lambda_{22}] \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

$$+ \frac{m}{\Delta t}[\lambda_{11}\ \lambda_{12}] \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$+ \int_0^{\Delta t} \{ \lambda_1 \text{Fmax}\ u_1 + \lambda_2 \text{Tmax}\ u_2 \}\ dt \ . \qquad (4.10)$$

For terms containing the controls two situations will be considered.

<u>No switching of controls</u>

If no switch (change of sign) occurs within the element, then

$$\text{sgn}(\lambda_{11}) = \text{sgn}(\lambda_{12}) \qquad (4.11.1)$$

and

$$\text{sgn}(\lambda_{21}) = \text{sgn}(\lambda_{22}) \qquad (4.11.2)$$

are true. Substituting equations (3.63), (4.6) and (4.9.3) into the first integral term in equation (4.10) gives

$$\int_o^{\Delta t} \{\lambda_1 \text{Fmax}\ u_1\}\ dt = -\frac{\text{Fmax}}{2} \Delta t\ (|\lambda_{11}| + |\lambda_{12}|) \ . \qquad (4.13)$$

Similarly the second integral term in equation (4.10) is found to be

$$\int_o^{\Delta t} \{\lambda_2 \text{Tmax}\ u_2\}\ dt = -\frac{\text{Tmax}}{2} \Delta t\ (|\lambda_{21}| + |\lambda_{22}|) \ . \qquad (4.14)$$

Switching of controls

If a switch in u_1 occurs within the element then

$$\mathrm{sgn}(\lambda_{11}) = - \mathrm{sgn}(\lambda_{12}) \ . \tag{4.15}$$

Let t_s be the time to switch from the start of the element in which the switching occurs. Then

$$\lambda_1(t_s) = 0 \tag{4.16}$$

or

$$N_1(t_s)\lambda_{11} + N_2(t_s)\lambda_{12} = 0 \tag{4.17}$$

or

$$(1 - \frac{t_s}{\Delta t})\lambda_{11} + (\frac{t_s}{\Delta t})\lambda_{12} = 0 \ . \tag{4.18}$$

Rearranging the above equation yields

$$t_s = \frac{\Delta t \ \lambda_{11}}{\lambda_{11} - \lambda_{12}} \ . \tag{4.19}$$

Therefore, for a switch

$$\int_0^{\Delta t} \{\lambda_1 \mathrm{Fmax} \ u_1 \} \ dt = - \int_0^{t_s} \{\lambda_1 \mathrm{Fmax} \ \mathrm{sgn}(\lambda_{11})\} \ dt - \int_{t_s}^{\Delta t} \{\lambda_1 \mathrm{Fmax} \ \mathrm{sgn}(\lambda_{12})\} \ dt \ . \tag{4.20}$$

Substituting equation (4.19) into (4.20) above and simplifying gives

$$\int_0^{\Delta t} \{\lambda_1 \mathrm{Fmax} \ u_1 \} \ dt = -\frac{\mathrm{Fmax}}{2} \ \mathrm{sgn}(\lambda_{11}) \ \Delta t \ \frac{(\lambda_{11}^2 + \lambda_{12}^2)}{(\lambda_{11} - \lambda_{12})} \ . \tag{4.21}$$

For switch in u_2 within the element

$$\mathrm{sgn}(\lambda_{21}) = - \mathrm{sgn}(\lambda_{22}) \ . \tag{4.22}$$

Following the same steps as in the case of u_1 switch gives

$$t_s = \frac{\Delta t \ \lambda_{21}}{\lambda_{21} - \lambda_{22}} \tag{4.23}$$

and

$$\int_0^{\Delta t} \{\lambda_2 \mathrm{Tmax}\ u_2\}\ dt = -\frac{\mathrm{Tmax}}{2}\ \mathrm{sgn}(\lambda_{21})\ \Delta t\ \frac{(\lambda_{21}^2 + \lambda_{22}^2)}{(\lambda_{21} - \lambda_{22})} . \tag{4.24}$$

Element equations from a Variational principle

The finite element solution to the problem involves picking the values of ϕ_i(consisting of r, θ, λ_1, λ_2) where i goes from 1 to p and p is equal to four times the number of nodes, and element length Δt, so as to make the functional $J_a(\phi, \Delta t)$ stationary. To make $J_a(\phi, \Delta t)$ stationary with respect to ϕ_i and Δt the fundamental theorem of variational calculus requires that

$$\delta J_a(\phi,\Delta t) = \frac{\partial J_a}{\partial \Delta t}\ \delta\Delta t + \sum_{i=1}^{p} \frac{\partial J_a}{\partial \phi_i}\ \delta\phi_i = 0 . \tag{4.25}$$

Since the $\delta\phi_i$'s and Δt are independant, equation (4.25) can hold only if

$$\frac{\partial J_a}{\partial \phi_i} = 0 \tag{4.26.1}$$

and

$$\frac{\partial J_a}{\partial \Delta t} = 0 . \tag{4.26.2}$$

Therefore, J_a in equation (4.10) is differentiated with respect to the nodal unknowns, r_1, r_2, θ_1, θ_2, λ_{11}, λ_{12} λ_{21}, λ_{22}, and with respect to

time Δt, to get the element equations. Differentiating J_a with respect to r_1, r_2 gives

$$\frac{\partial J_a}{\partial \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}} = \frac{m}{6\Delta t}[\theta_1 \ \theta_2] \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \lambda_{11} \\ \lambda_{12} \end{bmatrix}$$

$$+ \frac{m}{3\Delta t}[\lambda_{21} \ \lambda_{22}] \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$+ \frac{m}{\Delta t} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{11} \\ \lambda_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \tag{4.27}$$

Let

$$\mathrm{cm13} = \frac{m}{6\Delta t} (\theta_1 - \theta_2)^2 , \tag{4.28}$$

$$\mathrm{cm11} = \frac{m}{3\Delta t} (\theta_1 - \theta_2)(\lambda_{21} - \lambda_{22}) , \tag{4.29}$$

and

$$\mathrm{cm31} = \frac{m}{\Delta t} . \tag{4.30}$$

Substituting equations (4.28), (4.29), (4.30) into (4.27) gives

$$\frac{\partial J_a}{\partial \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}} = cm13 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \lambda_{11} \\ \lambda_{12} \end{bmatrix} + cm11 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$+ cm31 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{11} \\ \lambda_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.31}$$

Let
$$cm22 = \frac{m}{3\Delta t}(\lambda_{11}[2r_1 + r_2] + \lambda_{12}[r_1 + 2r_2] \ , \tag{4.32}$$

and
$$cm24 = \frac{m}{3\Delta t}(r_1^2 + r_1 r_2 + r_2^2) \ . \tag{4.33}$$

Differentiating J_a with respect to θ_1 and θ_2 plus substituting cm22 and cm24 into the expression gives

$$\frac{\partial J_a}{\partial \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}} = cm22 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} + cm24 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{21} \\ \lambda_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \tag{4.34}$$

Differentiating J_a with respect to λ_{11} and λ_{12}, and substituting cm13 and cm31 into the expression gives

$$\frac{\partial J_a}{\partial \begin{bmatrix}\lambda_{11}\\ \lambda_{12}\end{bmatrix}} = \text{cm13}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}r_1\\ r_2\end{bmatrix} + \text{cm31}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}r_1\\ r_2\end{bmatrix}$$

$$+ \frac{\partial}{\partial \begin{bmatrix}\lambda_{11}\\ \lambda_{12}\end{bmatrix}} \int_0^{\Delta t} \{\lambda_1 F\text{max}\ u_1\}\ dt = \begin{bmatrix}0\\ 0\end{bmatrix}. \tag{4.35}$$

For the last term of equation (4.35) there exist two situations depending upon the occurance of a switch in λ_1. If a switch does not occur then λ_1 does not change sign. Differentiating equation (4.13) with respect to λ_{11}, λ_{12} gives

$$\frac{\partial}{\partial \begin{bmatrix}\lambda_{11}\\ \lambda_{12}\end{bmatrix}} \int_0^{\Delta t} \{\lambda_1 F\text{max}\ u_1\}\ dt = -\frac{F\text{max}}{2}\ \Delta t\ \text{sgn}(\lambda_{11}) \begin{bmatrix}1\\ 1\end{bmatrix} \tag{4.36}$$

If a switch occurs then λ_1 will pass through zero. Differentiating equation (4.21) with respect to λ_{11} and λ_{12} gives

$$\frac{\partial}{\partial\ \lambda_{11}} \int_0^{\Delta t} \{\lambda_1 F\text{max}\ u_1\}\ dt = -\ \frac{F\text{max}}{2}\ \Delta t\ \text{sgn}(\lambda_{11})$$

$$\left[\frac{\lambda_{11}}{\lambda_{11} - \lambda_{12}} - \frac{(\lambda_{11}^2 + \lambda_{12}^2)}{2(\lambda_{11} - \lambda_{12})^2}\right] \tag{4.37.1}$$

and $$\frac{\partial}{\partial \lambda_{12}} \int_0^{\Delta t} \{\lambda_1 \text{Fmax}\ u_1\}\ dt = -\frac{\text{Fmax}}{2} \Delta t\ \text{sgn}(\lambda_{11})$$

$$\left[\frac{\lambda_{12}}{\lambda_{11} - \lambda_{12}} + \frac{(\lambda_{11}^2 + \lambda_{12}^2)}{2(\lambda_{11}-\lambda_{12})^2}\right]. \quad (4.37.2)$$

Differentiating J_a with respect to λ_{21}, λ_{22} and substituting cm24 into the expression gives

$$\frac{\partial J_a}{\partial \begin{bmatrix} \lambda_{21} \\ \lambda_{22} \end{bmatrix}} = \text{cm24} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

$$+ \frac{\partial}{\partial \begin{bmatrix} \lambda_{21} \\ \lambda_{22} \end{bmatrix}} \int_0^{\Delta t} \{\lambda_2 \text{Tmax}\ u_2\}\ dt = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (4.38)$$

For the last term in equation (4.38) there exist two situations depending upon the occurrence of a switch in λ_2. For the situation of no switch, differentiating equation (4.14) with respect to λ_{21} and λ_{22} gives

$$\frac{\partial}{\partial \begin{bmatrix} \lambda_{21} \\ \lambda_{22} \end{bmatrix}} \int_0^{\Delta t} \{\lambda_2 \text{Tmax}\ u_2\}\ dt = -\frac{\text{Tmax}}{2} \Delta t\ \text{sgn}(\lambda_{21}) \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (4.39)$$

For the situation where a switch occurs, differentiating equation (4.24) with respect to λ_{21} and λ_{22} gives

$$\frac{\partial}{\partial \lambda_{21}} \int_0^{\Delta t} \{\lambda_2 Tmax\ u_2\}\ dt = -\frac{Tmax}{2}\ \Delta t\ sgn(\lambda_{21})$$

$$\left[\frac{\lambda_{21}}{\lambda_{21} - \lambda_{22}} - \frac{(\lambda_{21}^2 + \lambda_{22}^2)}{2(\lambda_{21} - \lambda_{22})^2}\right] \quad (4.40.1)$$

and

$$\frac{\partial}{\partial \lambda_{22}} \int_0^{\Delta t} \{\lambda_2 Tmax\ u_2\}\ dt = -\frac{Tmax}{2}\ \Delta t\ sgn(\lambda_{21})$$

$$\left[\frac{\lambda_{22}}{\lambda_{21} - \lambda_{22}} + \frac{(\lambda_{21}^2 + \lambda_{22}^2)}{2(\lambda_{21} - \lambda_{22})^2}\right]. \quad (4.40.2)$$

Derivation of the transversality equation.

Differentiating J_a in equation (4.10) with respect to time Δt and simplifying gives

$$\frac{\partial J_a}{\partial \Delta t} = 1 - \frac{1}{\Delta t}[\ cm22\ \frac{3\Delta t}{m}\ cm13 + cm24\ \frac{3\Delta t}{m}\ cm11 + cm31(r_1 - r_2)(\lambda_{11} - \lambda_{12})]$$

$$+ \frac{\partial}{\partial \Delta t} \int_0^{\Delta t} \{\lambda_1 Fmax\ u_1 + \lambda_2 Tmax\ u_2\}\ dt\ . \quad (4.41)$$

There are two situations for the integral terms. If no switch occurs, then

$$\frac{\partial}{\partial \Delta t} \int_0^{\Delta t} \{\lambda_1 Fmax\ u_1\}\ dt = -\frac{Fmax}{2}\ (|\ \lambda_{11}| + |\ \lambda_{12}|) \quad (4.42)$$

and

$$\frac{\partial}{\partial \Delta t} \int_0^{\Delta t} \{\lambda_2 Tmax\ u_2\}\ dt = -\frac{Tmax}{2}\ (|\ \lambda_{21}| + |\ \lambda_{22}|)\ . \quad (4.43)$$

If a switch on u_1 occurs, then

$$\frac{\partial}{\partial \Delta t} \int_0^{\Delta t} \{\lambda_1 Fmax\ u_1\ \}\ dt = -\frac{Fmax}{2}\ sgn(\lambda_{11}) \frac{(\lambda_{11}^2 + \lambda_{12}^2)}{(\lambda_{11} - \lambda_{12})} \quad . \tag{4.44}$$

If a switch on u_2 occurs, then

$$\frac{\partial}{\partial \Delta t} \int_0^{\Delta t} \{\lambda_2 Tmax\ u_2\ \}\ dt = -\frac{Tmax}{2}\ sgn(\lambda_{21}) \frac{(\lambda_{21}^2 + \lambda_{22}^2)}{(\lambda_{21} - \lambda_{22})} \quad . \tag{4.45}$$

The nine element equations derived above, (4.31), (4.34), (4.35), (4.38) and (4.41) are assembled to give the 9 x 9 element matrix [A] as shown in Figure 4.2. The nodal unknowns, r_1, r_2, θ_1, θ_2, λ_{11}, λ_{12}, λ_{21}, λ_{22} ,and Δt constitute the vector $\mathbf{x}$. The element sub-matrices, M(i,j), are two by two matrices for values of i and j ranging from one to four. The submatrices M(i,5) are two by one vectors while the submatrices M(5,i) are one by two row vectors. The submatrix M(5,5) is a scalar. The element equation can be stated as

$$[\ A\ ]\ \mathbf{x} = f(\mathbf{x}) = 0\ . \tag{4.46}$$

To solve the system of nonlinear equations the Newton-Raphson method will be used. The nonlinear equations are linearized by using a Taylor series expansion about the true solution $f(\mathbf{x}^*)$. Guesses on the unknown variables can be expressed as

$$\mathbf{x}_i = \mathbf{x}^* + \Delta \mathbf{x}_i , \tag{4.47}$$

where i is the iteration number and $\Delta \mathbf{x}_i$ is the vector of deviations from the solution vector $\mathbf{x}^*$.

$$\underbrace{\begin{bmatrix} M(1,1) & M(1,2) & M(1,3) & M(1,4) & M(1,5) \\ M(2,1) & M(2,2) & M(2,3) & M(2,4) & M(2,5) \\ M(3,1) & M(3,2) & M(3,3) & M(3,4) & M(3,5) \\ M(4,1) & M(4,2) & M(4,3) & M(4,4) & M(4,5) \\ M(5,1) & M(5,2) & M(5,3) & M(5,4) & M(5,5) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} R_1 \quad R_2 \\ \theta_1 \quad \theta_2 \\ L_{11} \quad L_{12} \\ L_{21} \quad L_{22} \\ DT \end{bmatrix}}_{X} = \begin{bmatrix} 0 \end{bmatrix}$$

FIGURE 4.2 ELEMENT MATRIX EQUATION

Considering the first two terms of the Taylor series expansion gives

$$f\,(x_1^* + \Delta x_1,\ x_2^* + \Delta x_2, \ldots, x_p^* + \Delta x_p) = f\,(x_1^*,\ x_2^*, \ldots, x_p^*) + \frac{\partial f}{\partial x_1}\,\Delta x_1$$

$$+ \frac{\partial f}{\partial x_2}\,\Delta x_2 + \ldots + \frac{\partial f}{\partial x_p}\,\Delta x_p \qquad (4.48)$$

where f is a vector of order p for a system of p equations. From equation (4.46)

$$f(\overset{*}{x}) = 0\ . \qquad (4.49)$$

Therefore, equation (4.48) reduces to

$$f(x) = \frac{\partial f}{\partial x}\,\Delta x\ . \qquad (4.50)$$

The Jacobian [J] is defined as

$$[\ J\ ] = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_p} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_p} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_p}{\partial x_1} & \frac{\partial f_p}{\partial x_2} & \cdots & \frac{\partial f_p}{\partial x_p} \end{bmatrix}. \qquad (4.51)$$

Therefore equation (4.50) can be written as

$$f(x) = [\ J\ ]\ \Delta x\ . \qquad (4.52)$$

Equation (4.50) is solved iteratively and the guesses are updated each iteration until the convergence criterion

$$\mathrm{CONV} = \left[\sum_{i=1}^{n} \frac{(\Delta x_i)^2}{(x_i)^2} \right]^{0.5} < 1.0\mathrm{E}{-10} \tag{4.53}$$

is satisfied.

<u>Evualation of the Jacobian</u>

The Jacobian is a 9 x 9 matrix that is defined by several sub-matrices as shown in Figure 4.3. These sub-matrices are presented in the following development.

From eqnations (4.46) and (4.51)

$$J(1,1) = \frac{\partial^2 J_a}{\partial^2 \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}} = \mathrm{cm11} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{4.54}$$

and

$$J(1,2) = \frac{\partial^2 J_a}{\partial \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \partial \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}} = \mathrm{cj12} \begin{bmatrix} (1,3) & -(1,3) \\ (2,3) & -(2,3) \end{bmatrix} \tag{4.55.1}$$

where

$$\mathrm{cj12} = \frac{m}{3\Delta t}, \tag{4.55.2}$$

$$(1,3) = (\lambda_{21} - \lambda_{22})(2r_1 + r_2) + (\theta_1 - \theta_2)(2\lambda_{11} + \lambda_{12}), \tag{4.55.3}$$

and

$$(2,3) = (\lambda_{21} - \lambda_{22})(r_1 + 2r_2) + (\theta_1 - \theta_2)(\lambda_{11} + 2\lambda_{12}). \tag{4.55.4}$$

$$\underbrace{\begin{bmatrix} J(1,1) & J(1,2) & J(1,3) & J(1,4) & J(1,5) \\ J(2,1) & J(2,2) & J(2,3) & J(2,4) & J(2,5) \\ J(3,1) & J(3,2) & J(3,3) & J(3,4) & J(3,5) \\ J(4,1) & J(4,2) & J(4,3) & J(4,4) & J(4,5) \\ J(5,1) & J(5,2) & J(5,3) & J(5,4) & J(5,5) \end{bmatrix}}_{J} \underbrace{\begin{bmatrix} DR_1 \\ DR_2 \\ D\theta_1 \\ D\theta_2 \\ DL_{11} \\ DL_{12} \\ DL_{21} \\ DL_{22} \\ DDT \end{bmatrix}}_{DX} = \begin{bmatrix} 0 \end{bmatrix}$$

FIGURE 4.3 ELEMENT JACOBIAN EQUATION

The (1,3) and (1,4) submatrices are

$$J(1,3) = \frac{\partial^2 J_a}{\partial \begin{bmatrix} \lambda_{11} \\ \lambda_{12} \end{bmatrix} \partial \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}} = cm13 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + cm31 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{4.56}$$

and

$$J(1,4) = \frac{\partial^2 J_a}{\partial \begin{bmatrix} \lambda_{21} \\ \lambda_{22} \end{bmatrix} \partial \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}} = cj14 \begin{bmatrix} (1,7) & -(1,7) \\ (2,7) & -(2,7) \end{bmatrix} \tag{4.57.1}$$

where

$$cj14 = \frac{m}{3\Delta t} (\theta_1 - \theta_2) \, , \tag{4.57.2}$$

$$(1,7) = (2r_1 + r_2) \quad , \tag{4.57.3}$$

and

$$(2,7) = (r_1 + 2r_2) \quad . \tag{4.57.4}$$

The (1,5) submatrix is

$$J(1,5) = \frac{\partial^2 J_a}{\partial \Delta t \quad \partial \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}} = - \frac{1}{\Delta t} \Bigg[\; cm11 \begin{bmatrix} 2r_1 + r_2 \\ r_1 + 2r_2 \end{bmatrix} + cm13 \begin{bmatrix} 2\lambda_{11} + \lambda_{12} \\ \lambda_{11} + 2\lambda_{12} \end{bmatrix}$$

$$+ \, cm31 \begin{bmatrix} \lambda_{11} - \lambda_{12} \\ -\lambda_{11} + \lambda_{12} \end{bmatrix} \; \Bigg] \; . \tag{4.58}$$

Since the Jacobian is symmetric we have the condition

$$J(2,1) = \frac{\partial^2 J_a}{\partial \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \partial \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}} = [\; J(1,2) \;]^T \; . \tag{4.59}$$

The other terms from the second row of submatrices in the Jacobian are

$$J(2,2) = \frac{\partial^2 J_a}{\partial^2 \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}} = cm22 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \tag{4.60}$$

$$J(2,3) = \frac{\partial^2 J_a}{\partial \begin{bmatrix}\lambda_{11}\\ \lambda_{12}\end{bmatrix} \partial \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}} = cj14 \begin{bmatrix} (1,7) & (2,7) \\ -(1,7) & -(2,7) \end{bmatrix}, \tag{4.61}$$

$$J(2,4) = \frac{\partial^2 J_a}{\partial \begin{bmatrix}\lambda_{21}\\ \lambda_{22}\end{bmatrix} \partial \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}} = cm24 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \tag{4.62}$$

and $$J(2,5) = \frac{\partial^2 J_a}{\partial \Delta t \quad \partial \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}} = -\frac{1}{\Delta t}[cm22 \begin{bmatrix} \theta_1 - \theta_2 \\ -\theta_1 + \theta_2 \end{bmatrix} + cm24 \begin{bmatrix} \lambda_{21} - \lambda_{22} \\ -\lambda_{21} + \lambda_{22} \end{bmatrix}. \tag{4.63}$$

By symmetry we have

$$J(3,1) = \frac{\partial^2 J_a}{\partial \begin{bmatrix}r_1\\ r_2\end{bmatrix} \partial \begin{bmatrix}\lambda_{11}\\ \lambda_{12}\end{bmatrix}} = [\ J(1,3)\ ]^T \tag{4.64}$$

and $$J(3,2) = \frac{\partial^2 J_a}{\partial \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \partial \begin{bmatrix} \lambda_{11} \\ \lambda_{12} \end{bmatrix}} = [\ J(2,3)\ ]^T \ . \tag{4.65}$$

There are two separate expressions for J(3,3) given by

$$J(3,3) = \frac{\partial^2 J_a}{\partial^2 \begin{bmatrix} \lambda_{11} \\ \lambda_{12} \end{bmatrix}} \ . \tag{4.66.1}$$

For no switch on u_1 we have

$$J(3,3) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \ . \tag{4.66.2}$$

For switch on u_1 we get the result

$$J(3,3) = cj33 \begin{bmatrix} \lambda_{12}^2 & \lambda_{11}\lambda_{12} \\ \lambda_{11}\lambda_{12} & \lambda_{11}^2 \end{bmatrix} \tag{4.66.3}$$

where $$cj33 = -\frac{2\ Fmax\ \Delta t}{(\lambda_{11} - \lambda_{12})^3}\ sgn(\lambda_{11}) \ . \tag{4.66.4}$$

The last two terms in row 3 are the submatrices

$$J(3,4) = \frac{\partial^2 J_a}{\partial \begin{bmatrix}\lambda_{21}\\ \lambda_{22}\end{bmatrix} \partial \begin{bmatrix}\lambda_{11}\\ \lambda_{12}\end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{4.67}$$

and

$$J(3,5) = \frac{\partial^2 J_a}{\partial \Delta t \quad \partial \begin{bmatrix}\lambda_{11}\\ \lambda_{12}\end{bmatrix}} = -\frac{1}{\Delta t} \begin{bmatrix} (5,9) \\ (6,9) \end{bmatrix} \tag{4.68.1}$$

where the quantities (5,9) and (6,9) for no switch on u_1 are given by

$$(5,9) = (2cm13 + cm31)r_1 + (cm13 - cm31)r_2 - \frac{Fmax}{2} \Delta t \; sgn(\lambda_{11}) \tag{4.68.2}$$

and

$$(6,9) = (cm13 - cm31)r_1 + (2cm13 + cm31)r_2 - \frac{Fmax}{2} \Delta t \; sgn(\lambda_{11}) \tag{4.68.3}$$

while for switch on u_1 we get

$$(5,9) = (2cm13 + cm31)r_1 + (cm13 - cm31)r_2 - \frac{Fmax}{2} \Delta t$$

$$sgn(\lambda_{11})\left[\frac{\lambda_{11}}{(\lambda_{11} - \lambda_{12})} - \frac{(\lambda_{11}^2 + \lambda_{12}^2)}{2(\lambda_{11} - \lambda_{12})^2}\right] \tag{4.68.4}$$

and

$$(6,9) = (cm13 - cm31)r_1 + (2cm13 + cm31)r_2 - \frac{Fmax}{2} \Delta t$$

$$sgn(\lambda_{11})\left[\frac{\lambda_{12}}{(\lambda_{11} - \lambda_{12})} + \frac{\lambda_{11}^2 + \lambda_{12}^2}{2(\lambda_{11} - \lambda_{12})^2}\right] . \tag{4.68.5}$$

For the fourth row of submatrices we have by symmetry

$$J(4,1) = \frac{\partial^2 J_a}{\partial \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \partial \begin{bmatrix} \lambda_{21} \\ \lambda_{22} \end{bmatrix}} = [\ J(1,4)\ ]^T , \tag{4.69}$$

$$J(4,2) = \frac{\partial^2 J_a}{\partial \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \partial \begin{bmatrix} \lambda_{21} \\ \lambda_{22} \end{bmatrix}} = [\ J(2,4)\ ]^T , \tag{4.70}$$

and

$$J(4,3) = \frac{\partial^2 J_a}{\partial \begin{bmatrix} \lambda_{11} \\ \lambda_{12} \end{bmatrix} \partial \begin{bmatrix} \lambda_{21} \\ \lambda_{22} \end{bmatrix}} = [\ J(3,4)\ ]^T . \tag{4.71}$$

There are two separate expressions for J(4,4) given by

$$J(4,4) = \frac{\partial^2 J_a}{\partial^2 \begin{bmatrix} \lambda_{21} \\ \lambda_{22} \end{bmatrix}} . \tag{4.72.1}$$

For no switch on u_2 we have

$$J(4,4) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} . \tag{4.72.2}$$

For switch on u_2 we have the result

$$J(4,4) = cj44 \begin{bmatrix} \lambda_{22}^2 & -\lambda_{21}\lambda_{22} \\ -\lambda_{21}\lambda_{22} & \lambda_{21}^2 \end{bmatrix} \tag{4.72.3}$$

where

$$cj44 = -\frac{2\ Tmax\ \Delta t}{(\lambda_{21} - \lambda_{22})^3}\ sgn(\lambda_{21}) \ . \tag{4.72.4}$$

The last term in the fourth row of submatrices is

$$J(4,5) = \frac{\partial^2 J_a}{\partial \Delta t \ \ \partial \begin{bmatrix} \lambda_{21} \\ \lambda_{22} \end{bmatrix}} = -\frac{1}{\Delta t} \begin{bmatrix} (7,9) \\ (8,9) \end{bmatrix} \tag{4.73.1}$$

where if no switch on u_2 occurs the quantities (7,9) and (8,9) are given by

$$(7,9) = cm24\ \theta_1 - cm24\ \theta_2 - \frac{Tmax}{2}\ \Delta t\ sgn(\lambda_{21}) \tag{4.73.2}$$

and

$$(8,9) = -\ cm24\ \theta_1 + cm24\ \theta_2 - \frac{Tmax}{2}\ \Delta t\ sgn(\lambda_{21}) \ . \tag{4.73.3}$$

If a switch occurs on u_2 then (7,9) and (8,9) are given by

$$(7,9) = cm24\ \theta_1 - cm24\ \theta_2 - \frac{Tmax}{2}\ \Delta t\ sgn(\lambda_{21})$$

$$\left[\frac{\lambda_{21}}{(\lambda_{21} - \lambda_{22})} - \frac{(\lambda_{21}^2 + \lambda_{22}^2)}{2(\lambda_{21} - \lambda_{22})^2}\right] \tag{4.74.4}$$

and $$(8,9) = - cm24\ \theta_1 + cm24\ \theta_2 - \frac{Tmax}{2} \Delta t\ \mathrm{sgn}(\lambda_{21})$$

$$\left[\frac{\lambda_{22}}{(\lambda_{21} - \lambda_{22})} + \frac{\lambda_{21}^2 + \lambda_{22}^2}{2(\lambda_{21} - \lambda_{22})^2}\right] . \qquad (4.74.5)$$

For the last row of submatrices we have by symmetry

$$J(5,1) = \frac{\partial^2 J_a}{\partial \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \partial \Delta t} = [\ J(1,5)\ ]^T , \qquad (4.75)$$

$$J(5,2) = \frac{\partial^2 J_a}{\partial \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \partial \Delta t} = [\ J(2,5)\ ]^T , \qquad (4.76)$$

$$J(5,3) = \frac{\partial^2 J_a}{\partial \begin{bmatrix} \lambda_{11} \\ \lambda_{12} \end{bmatrix} \partial \Delta t} = [\ J(3,5)\ ]^T , \qquad (4.77)$$

and $$J(5,4) = \frac{\partial^2 J_a}{\partial \begin{bmatrix} \lambda_{21} \\ \lambda_{22} \end{bmatrix} \partial \Delta t} = [\ J(4,5)\ ]^T . \qquad (4.78)$$

The last term in the fifth row of the submatrices is given by

$$J(5,5) = \frac{\partial^2 J_a}{\partial^2 \Delta t} = \frac{2}{\Delta t^2} \left[\frac{3\Delta t}{m} cm22\ cm13 + \frac{3\Delta t}{m} cm24\ cm11\right.$$

$$\left. + cm31(r_1 - r_2)(\lambda_{11} - \lambda_{12})\right] . \qquad (4.79)$$

Assembly of element properties to obtain system equations

To solve for the unknowns in the whole solution region it is necessary to assemble or combine the element matrix equations to form the global equations governing the behavior over the entire problem domain. The basis for the assembly procedure stems from the fact that at a node where elements are interconnected, the values of the unknowns are the same for each element sharing that node. The assembly is performed by two routines. These are:

1. Node : Converts the nodal unknowns in the local (element) numbering scheme to the nodal unknowns in the global (system) numbering scheme (refer to Appendix II).
2. Build : Combines all the element unknowns to form the system unknowns (refer to Appendix II).

Application of boundary conditions and solution of system unknowns

Before the system equations can be solved, they must be modified to account for the boundary conditions of the problem, otherwise the system matrix will be singular. The application of the boundary conditions is performed in routine Solve (refer to Appendix II) where in addition to the boundary conditions on the states the transversality equation is applied on the Lagrangian multipliers λ_1 and λ_2 at both the initial and final time.

Application of transversality equation
as boundary conditions

Substituting equations (3.67) and (3.69) into the transversality equation (3.39) gives the equation

$$f(\mathbf{x}) = \text{Fmax}\ \text{sgn}(\lambda_1)\lambda_1 + \text{Tmax}\ \text{sgn}(\lambda_2)\lambda_2 - 1 = 0\ . \tag{4.80}$$

To apply equation (4.80) in the Newton-Raphson iteration it should be of the form

$$\frac{\partial f}{\partial \mathbf{x}} \Delta \mathbf{x} = 0\ . \tag{4.81}$$

Differentiating equation (4.80) with respect to λ_1 gives

$$\frac{\partial f}{\partial \lambda_1} = \text{Fmax}\ \text{sgn}(\lambda_1)\ . \tag{4.82}$$

Differentiating equation (4.80) with respect to λ_2 gives

$$\frac{\partial f}{\partial \lambda_2} = \text{Tmax}\ \text{sgn}(\lambda_2)\ . \tag{4.83}$$

Substituting equations (4.82) and (4.83) into equation (4.81) gives

$$\text{Fmax}\ \text{sgn}(\lambda_1)\Delta\lambda_1 + \text{Tmax}\ \text{sgn}(\lambda_2)\Delta\lambda_2 = f(\mathbf{x}_i) \tag{4.84}$$

where $f(\mathbf{x}^*)$ is equal to zero. The right side vector elements $f(\mathbf{x}_i)$ at t_o and t_f are modified in the MATRX subroutine while the left hand side of equation (4.84) is implemented in the SOLVE subroutine (refer Appendix II).

In chapter 5 the results from the continuous time and the discrete time simulations are presented. The discrete case is compared to the

continuous case. The effect of varying the grid density, on the final time is examined. Recommendations for further study are also presented.

CHAPTER 5

RESULTS AND RECOMMENDATIONS

Introduction

In this paper two methods of determining the minimum time control for the r-theta manipulator, the continuous time method and the discrete time method were investigated. Two formulations, the first order and the second order, were used in the continuous time method. In the discrete time method the second order formulation was used. The simulations were written in Fortran 77 and implemented on an IBM 370 mainframe as well as a HARRIS H-800 computer. The reason for using two computer systems was the availability of different minimization routines on the two systems.

Continuous Time Method

Four IMSL routines [18] on the IBM 370 were used in the continuous time simulation. These include DVERK (a differential equation solver), LEQT1F (a linear equations solver) and two minimization routines, ZXMIN (a Quasi-Newton Method), and ZXCGR (a conjugate-gradient method).

The continuous time problem required a large number of iterations in order to get good initial guesses on the unknowns (initial values on λ_1, λ_2, λ_3, λ_4 and final time t_f). A combination of the conjugate-gradient method and Newton's method was used. However, if the

initial guesses were not close to the optimum values the routines diverged. This difficulty is caused by the state and multiplier equations being very sensitive to the values of the Lagrangian multipliers. If the guesses are such that they do not cause switches in the multipliers then the Jacobian becomes singular. It is then necessary to come up with guesses on the multipliers at the initial time that bring about the correct number of switches in order for the minimization routine to converge.

The minimum time problem is solved for three cases.

Case 1

$$r(t_o) = 1.0 \,, \tag{5.1}$$

$$r(t_f) = 1.0 \,, \tag{5.2}$$

$$\theta(t_o) = 0.0 \,, \tag{5.3}$$

and

$$\theta(t_f) = 1.5708 \,. \tag{5.4}$$

Case 2

$$r(t_o) = 1.5 \,, \tag{5.5}$$

$$r(t_f) = 1.5 \,, \tag{5.6}$$

$$\theta(t_o) = 0.0 \,, \tag{5.7}$$

and

$$\theta(t_f) = 1.208 \,. \tag{5.8}$$

Case 3

$$r(t_o) = 1.2 \,, \tag{5.9}$$

$$r(t_f) = 1.2 \,, \tag{5.10}$$

$$\theta(t_o) = 0.0 \,, \tag{5.11}$$

and $$\theta(t_f) = 1.795 \ . \qquad (5.12)$$

The velocities $\dot{r}$ and $\dot{\theta}$ at both t_o and t_f are set to zero.

Results for the continuous time simulation for case 1 are shown in Figures 5.1 - 5.4.

Figure 5.1 shows the trajectory of the first joint variable θ and its first and second time derivatives. These variables are indicated by T, TD, and TDD respectively, in the figure.

Figure 5.2 shows the trajectory of the second joint variable r and its first and second time derivatives. These variables are indicated by R, RD, and RDD respectively, in the figure.

Figure 5.3 illustrates the trajectory of the Lagrangian multiplier λ_1, and its first and second time derivatives. These variables are indicated by L1, LD1, and LDD1 respectively, in the figure.

Figure 5.4 illustrates the trajectory of the Lagrangian multiplier λ_2, and its first and second time derivatives. These variables are indicated by L2, LD2, and LDD2 respectively, in the figure.

All the variables are seen to exhibit either even or odd symmetry about $t_f/2$. The second derivative curves are not smooth at points indicating the switchings of the bang bang controls.

Discrete Time Method

Three routines were used in the discrete time simulation. These include LEQT1F on the IBM 370, and two routines from Sandia Laboratories, MINA (a grid search minimization technique), and ODE (an

integration routine) which are available on the HARRIS. The finite element method also uses a combination of techniques. A grid search method is used to give a reasonably good guess on the final time t_f. This guess is then used in the finite element program to start the iterations on the nodal unknowns. The program then iterates until the convergence criterion has been satisfied. The guesses on the multpliers have to be of correct sign and must constitute a symmetrical path. The guesses on the joint variables r and θ must also constitute a symmetrical path. However the guesses on the magnitude of the multipliers can be quite far off, sometimes over a 100 percent. With a reasonable guess on the time t_f, convergence is achieved quite rapidly.

Results for the discrete time solution for the three cases are given in Table 5.1. The number of elements used for the three cases is twenty one. The effect of varying the grid density in the discrete time solution of Case 1 are presented in Table 5.2. Results for the first case are illustrated in Figures 5.5 - 5.8.

Figure 5.5 shows the trajectories of θ, and its first and second derivatives. Figure 5.6 shows the trajectories of r, and its first and second derivatives. Figure 5.7 illustrates the trajectories of λ_1, and its first and second derivatives. Figure 5.8 illustrates the trajectories of λ_2, and its first and second derivatives.

Conclusions and Recommendations

Comparing the two simulations it is seen that the discrete time simulation agrees very closely with the continuous case, except in the

TABLE 5.1

DISCRETE TIME SOLUTIONS FOR THE MINIMUM FINAL TIME

CASE	$r(t_o)$ (ft)	$r(t_f)$ (ft)	$\theta(t_o)$ (rad)	$\theta(t_f)$ (rad)	Final Time(t_f) (s)
1	1.0	1.0	0.0	1.5708	1.963
2	1.5	1.5	0.0	1.208	2.499
3	1.2	1.2	0.0	1.795	2.350

TABLE 5.2

EFFECT OF GRID DENSITY ON FINAL TIME FOR CASE 1

SIMULATION	NUMBER OF ELEMENTS	FINAL TIME(s)
continuous time		1.9644
discrete time	21	1.9629
	42	1.9640
	84	1.9643

second derivatives where the discrete time curves are smoother than the corresponding continuous curves. This is due to the linear interpolation function used in the formulation of the element equations. A closer agreement is obtained when the grid density is increased (refer Table 5.2). However this also introduces corresponding increases in storage space requirements.

The finite element method achieved the objective of applying a discrete time method to the solution of the minimum time control. The transversality equation that was enforced at both ends was critical in heading the iterations in the right direction. This equation is not only true at the initial and final time but also at the internal nodes.

The transversality equation at the internal nodes includes some velocity terms in addition to the terms in the equation at the initial and final time. An area of further investigation could be the application of the transversality equation at the internal nodes. The use of higher order interpolation functions for the formulation of the element equations can also be investigated. Since a linear interpolation function was used in this paper, it does not posses first derivative continuity.

This thesis has investigated the application of finite element methods to the solution of the minimum time problem. The deviation of the discrete time solution from the continuous time solution has been investigated and found to be reasonable. Some recomendations for further investigation in the area have been presented.

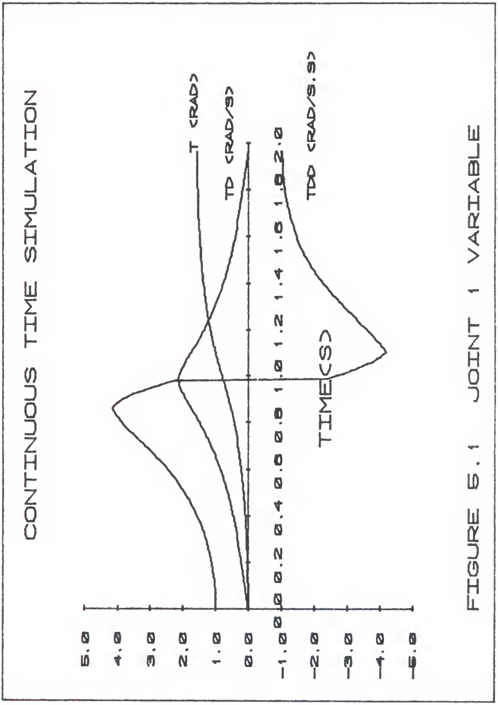

FIGURE 5.1 JOINT 1 VARIABLE

CONTINUOUS TIME SIMULATION

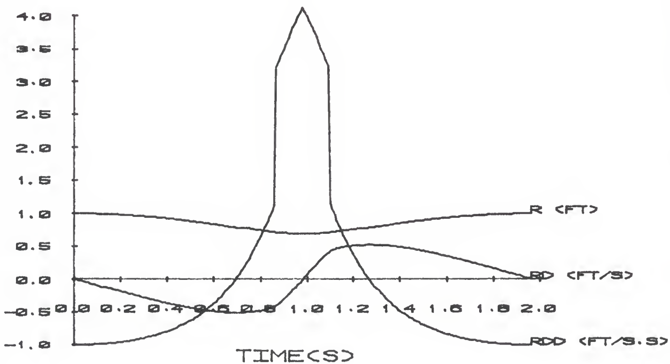

FIGURE 5.2 JOINT 2 VARIABLE

CONTINUOUS TIME SIMULATION

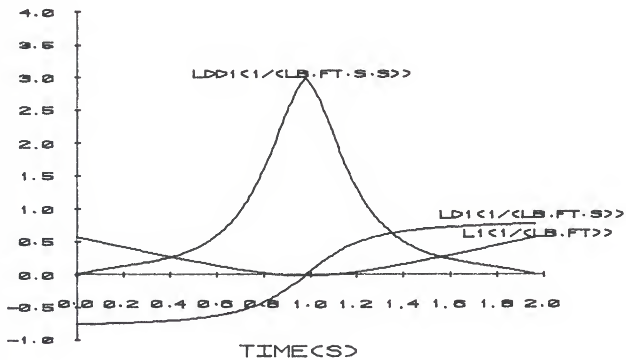

FIGURE 5.3 LAMBDA1

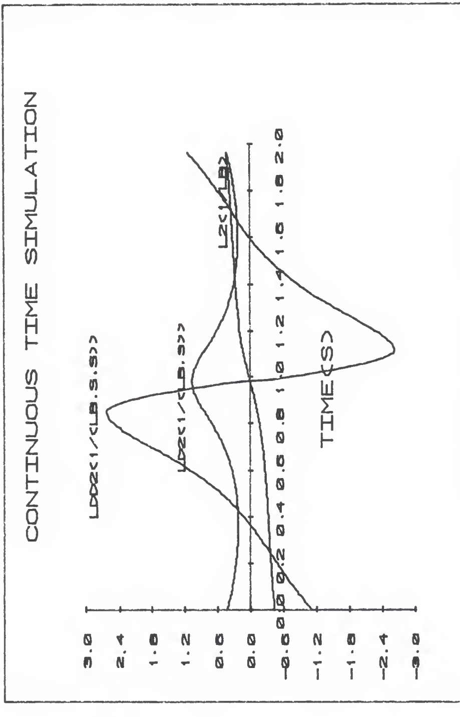

FIGURE 5.4 LAMBDA2

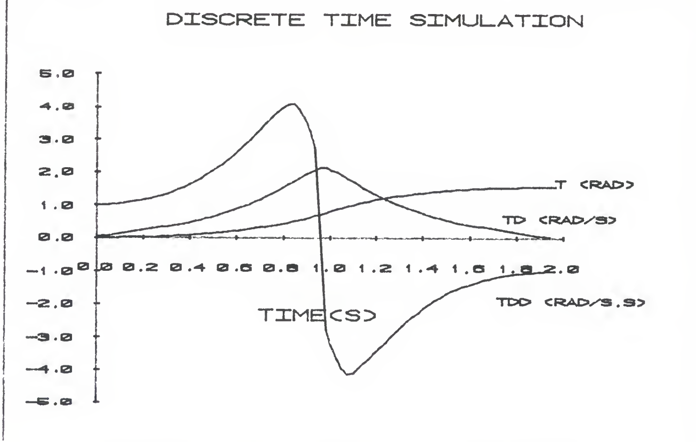

FIGURE 5.5 JOINT 1 VARIABLE

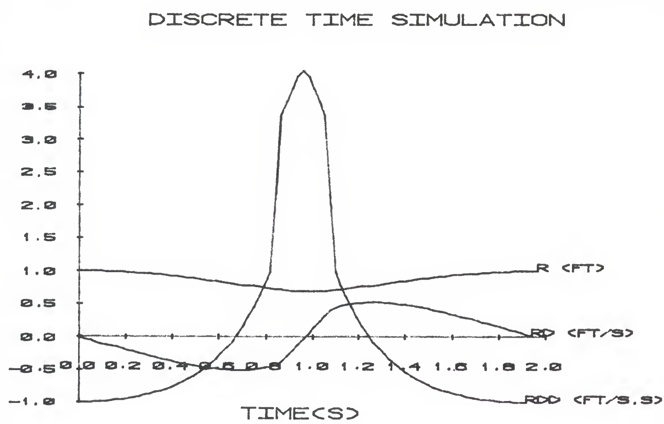

FIGURE 5.6 JOINT 2 VARIABLE

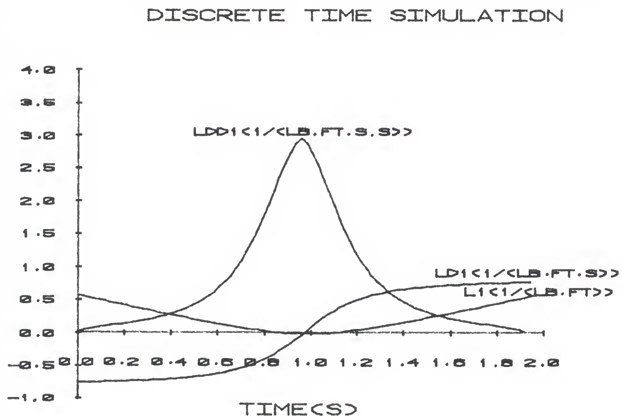

FIGURE 5.7 LAMBDA1

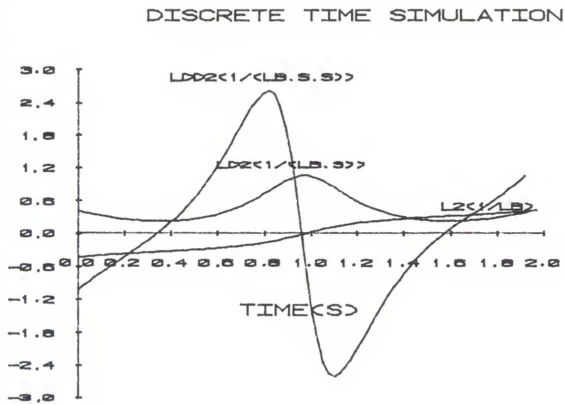

FIGURE 5.8 LAMBDA2

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APPENDIX I

DERIVATION OF THE MULTIPLIER EQUATIONS FOR THE SECOND ORDER FORMULATION

From equation (3.35) the performance index is defined as

$$J_a(x,\dot{x},u,\lambda,t) = \int_{t_o}^{t_f}\{1+ \lambda_1[\text{Fmax } u_1 - m\ddot{r} + mr\dot{\theta}^2 O$$

$$+ \lambda_2[\text{Tmax } u_2 - \frac{d}{dt}(mr^2\dot{\theta} \} \, dt \, . \qquad (a.1)$$

The variation of J_a with respect to r and θ is

$$\delta J_a = dt_f + \int_{t_o}^{t_f}\{- \lambda_1 m\delta\ddot{r} + m\lambda_1\dot{\theta}^2\delta r + \lambda_1 mr2\dot{\theta}\delta\dot{\theta}$$

$$- \lambda_2\delta \left(\frac{d}{dt} (mr^2\dot{\theta}) \right)\} \, dt \, . \qquad (a.2)$$

Integrating by parts and rearranging gives

$$\delta J_a = dt_f + (- \lambda_1 m\delta\dot{r} + \dot{\lambda}_1 \, m\delta r + mr2\dot{\theta}(\delta\theta)\lambda_1$$

$$- \lambda_2 m2r(\delta r)\dot{\theta} - \lambda_2 m\dot{r}^2 \, \delta\dot{\theta} + \dot{\lambda}_2 mr^2\delta\theta)\Big|_{t_o}^{t_f}$$

$$+ \int_{t_o}^{t_f}\{\delta r(- m\ddot{\lambda}_1 + m\lambda_1\dot{\theta}^2 + \dot{\lambda}_2 m2r\dot{\theta})$$

$$\delta\theta(- \frac{d}{dt} (2mr\dot{\theta}\lambda_1) - \frac{d}{dt} (\dot{\lambda}_2 mr^2))\} \, dt \, . \qquad (a.3)$$

For an extremal curve

$$\delta J_a = 0 \ . \tag{a.4}$$

Setting the integrand in (a.3) equal to zero gives the multiplier equations

$$- m\ddot{\lambda}_1 + m\lambda_1\dot{\theta}^2 + \dot{\lambda}_2 m2r\dot{\theta} = 0 \tag{a.5}$$

and

$$\frac{d}{dt}(2mr\dot{\theta}\lambda_1) + \frac{d}{dt}(\dot{\lambda}_2 mr^2) = 0 \ . \tag{a.6}$$

Therefore, equation (a.4) reduces to

$$\delta J_a = dt_f + (-\lambda_1 m\delta\dot{r} + \dot{\lambda}_1 \, m\delta r + mr2\dot{\theta}(\delta\theta)\lambda_1$$

$$- \lambda_2 m2r(\delta r)\dot{\theta} - \lambda_2 m\dot{r}^2 \, \delta\dot{\theta} + \dot{\lambda}_2 mr^2\delta\theta)\Big|_{t_o}^{t_f} \ . \tag{a.7}$$

The variation of a variable x at the final time is given by the relation

$$\delta x(t_f) = \delta x_f - \dot{x} \, dt_f \tag{a.8}$$

where δx_f is the variation of the final x, and $\dot{x}$ is the slope of x at time t_f. Using equation (3.8) in (3.7) gives

$$\delta J_a = dt_f + (\lambda_1 m \, \ddot{r} - \dot{\lambda}_1 m\dot{r} - mr2\dot{\theta}^2\lambda_1$$

$$+ \lambda_2 m2r\dot{\theta}\dot{r} + \lambda_2 mr^2 \, \ddot{\theta} - \dot{\lambda}_2 mr^2\dot{\theta})\Big|_{t=t_f} dt_f$$

$$- (- \lambda_1 m\delta\dot{r} + \dot{\lambda}_1 m\delta r + mr2\dot{\theta}\lambda_1\delta\theta$$

$$- \lambda_2 m2r\dot{\theta}\delta r - \lambda_2 mr^2\delta\dot{\theta} + \dot{\lambda}_2 mr^2\delta\theta)\Big|_{t=t_o} \ . \tag{a.9}$$

From equations (3.21) and (3.22)

$$\text{Fmax}\ \lambda_1\ u_1 = \lambda_1 m\ \ddot{r} - mr\dot{\theta}^2\ \lambda_1 \tag{a.10}$$

and

$$\text{Tmax}\ \lambda_2\ u_2 = \lambda_2 mr^2\ \ddot{\theta} + \lambda_2 m2r\dot{r}\dot{\theta}\quad . \tag{a.11}$$

The initial and final velocities $(\dot{r},\dot{\theta})$ are zero in the example and the initial and final states are specified. Substituting equations (a.10)and (a.11) into (a.9) together with the boundary conditions gives

$$1 + \text{Fmax}\ \lambda_1 u_1 + \text{Tmax}\ \lambda_2 u_2 = 0\quad . \tag{a.12}$$

The above equation is the transversality equation for the second order formulation.

APPENDIX II

FLOW CHART AND SUBROUTINES USED IN FINITE ELEMENT PROGRAM

START
↓
INITIALIZE VARIABLES
↓
MAKE GUESSES ON NODAL UNKNOWNS
↓
SUBROUTINE NODE
↓
START ITERATIONS
↓
SUBROUTINE SWITCH
↓
SUBROUTINE BUILD ⟷ SUBROUTINE MATRX
↓
SUBROUTINE SOLVE
↓
NO ← CONVERGED ?
↓
YES
↓
STOP

Figure A.1 Flow chart of Finite Element Program

```
C     ********************************************************************
C      The MAIN program initializez the variables, defines the initial
C     guesses on the unknowns,  calls the various subroutines and
C     controls the iterations.
C     ********************************************************************
C
C     ********************************************************************
C     Subroutine NODE is used to convert the local nodal unknowns into
C     the global ( or system) unknowns.
      SUBROUTINE NODE(NT,NTABLE,MAXELM)
C     ********************************************************************
      INTEGER NTABLE(MAXELM,8),NT,MAXELM,NELM,A1,A2,A3,A4,A5,A6,A7,A8
C
C     NELM = element number
C     NT = Number of elements
C      NTABLE = Table containing the global numbers for the element
C     unknowns
C     MAXELM = Maximum number of elements

      NELM=0
      DO 10 I=1,NT
        A1=4*(I-1)+1
        A2=A1+4
        A3=4*(I-1)+2
        A4=A3+4
        A5=4*(I-1)+3
```

```
      A6=A5+4
      A7=4*(I-1)+4
      A8=A7+4
C
      NELM=NELM+1
C
      NTABLE(NELM,1)=A1
      NTABLE(NELM,2)=A2
      NTABLE(NELM,3)=A3
      NTABLE(NELM,4)=A4
      NTABLE(NELM,5)=A5
      NTABLE(NELM,6)=A6
      NTABLE(NELM,7)=A7
      NTABLE(NELM,8)=A8
C
10    CONTINUE
C
      RETURN
      END
C     ****************************************************************
C     Subroutine SWITCH is used to determine if a switch of either
C     lambda1 or lambda2 or both, has occured within the element.  The
C     signs on the lambdas at the two nodes of the element are compared.
C     If the signs are different, a switch has occured within the
C     element.
C     ****************************************************************
```

```
C      *****************************************************************
C      *****************************************************************
C      Subroutine MATRX is used to set up the element matrix equations
C      as well as the element Jacobian.  The relations developed in
C      Chapter 4 are used in this routine.
C      *****************************************************************
C      *****************************************************************
C      Subroutine BUILD is used to assemble the element Jacobian into the
C      global Jacobian.
       SUBROUTINE BUILD(NT,NTABLE,ITABLE,JAC,RSV,SGNL1,SGNL2,TS1,TS2,GJ,M
      *AXRG,RSVG,MAXELM,MAT,R,THETA,LB1,LB2,MAXN,DT)
C      *****************************************************************
       REAL*8 JAC(9,9),RSV(9),SGNL1,SGNL2,TS1(MAXELM),TS2(MAXELM)
       REAL*8 GJ(MAXRG,MAXRG),RSVG(MAXRG),MAT(9,9)
       REAL*8 R(MAXN),THETA(MAXN),LB1(MAXN),LB2(MAXN),DT
       INTEGER NT,NTABLE(MAXELM,8),ITABLE(8),I1,I2
C      GJ is the global jacobian.
C      RSVG is the global right side vector
C      NT is the number of elements
C
C      ARRAYS ARE INITIALIZED
C
       DO 70 I=1,4*NT+5
         DO 80 J=1,4*NT+5
           GJ(I,J)=0.0
80       CONTINUE
```

```
          RSVG(I)=0.0
70     CONTINUE
C
C      THE GLOBAL JACOBIAN  AND GLOBAL RIGHT SIDE VECTOR ARE ASSEMBLED
C
       DO 10 I1=1,NT
         DO 20 J=1,8
           ITABLE(J)=NTABLE(I1,J)
20       CONTINUE
C
C        THE ELEMENT MATRIX,JACOBIAN AND RIGHT SIDE VECTOR ARE OBTAINED
C
       CALL MATRX(R,MAXN,THETA,LB1,LB2,DT,NT,SGNL1,SGNL2,TS1,TS2,MAXELM,M
     *AT,RSV,JAC,I1)

C      The element jacobians and the element right side vectors are
C       assembled into the global Jacobian and the global right side
C       vector here.

       DO 30 I2=1,8
         DO 40 J=1,8
             GJ(ITABLE(I2),ITABLE(J))=JAC(I2,J)+GJ(ITABLE(I2),ITABLE(J))
40       CONTINUE
         GJ(ITABLE(I2),4*NT+5)=JAC(I2,9)+GJ(ITABLE(I2),4*NT+5)
         RSVG(ITABLE(I2))=RSV(I2)+RSVG(ITABLE(I2))
30     CONTINUE
```

```
      DO 95 J=1,8
        GJ(4*NT+5,ITABLE(J))=JAC(9,J)+GJ(4*NT+5,ITABLE(J))
95    CONTINUE
      GJ(4*NT+5,4*NT+5)=JAC(9,9)+GJ(4*NT+5,4*NT+5)
      RSVG(4*NT+5)=RSV(9)+RSVG(4*NT+5)
C
10    CONTINUE


      RETURN
      END
C     ******************************************************************
C      Subroutine SOLVE is used to apply the boundary conditions and the
C     transversality equation on the global jacobian.  The system of
C     equations are solved by the IMSL routine LEQT1F (gaussian
C     elimination technique) and the guesses are updated.  The
C     convergence criteria is also computed here.
      SUBROUTINE SOLV(GJ,RSVG,MAXRG,NT,MRK,WK,R,THETA,LB1,LB2,MAXN,CONV,
     *DT,NTABLE,MAXELM,ITABLE,K)
C     ******************************************************************
      REAL*8 GJ(MAXRG,MAXRG),RSVG(MAXRG),WK(MRK),PI,CONVN,CONVD
      REAL*8 R(MAXN),THETA(MAXN),LB1(MAXN),LB2(MAXN),DT,CONV,A,B,C
      INTEGER M1,N2,IA,IDGT,IER,NTABLE(MAXELM,8),ITABLE(8),K
C     GJ is the global Jacobian
C     RSVG is the global right side vector
C     NT is the number of elements.
C     R is the Joint 2 variable
```

```
C     Theta is the joint 1 variable
C     LB1 is lambda1
C     LB2 is lambda2
C     DT is the length of element
C     CONV is the convergence criteria
C
      CONVN= 0.0
      CONVD= 0.0
      CONV= 0.0
C
C     BOUNDARY CONDITIONS ARE SPECIFIED
C
      DO 60 I=1,3
        DO 60 J=1,4*NT+5
          GJ(I,J)= 0.0
          GJ(4*NT+I,J)= 0.0
60      CONTINUE
      DO 65 I=1,2
        RSVG(I)= 0.0
        RSVG(4*NT+I)= 0.0
65    CONTINUE
C
      DO 68 I=1,2
        GJ(I,I)= 1.0
        GJ(4*NT+I,4*NT+I)= 1.0
68    CONTINUE
```

```
C
C     TRANSVERSALITY EQUATION APPLIED AT THE INITIAL AND FINAL TIME
C     known data
C     Fmax = 1
C     Tmax = 1
C     m = 1
C     at t = t    sgn(LB1 ) = 1
              o
C                 sgn(LB2) = -1
C     at t = t    sgn(LB1 ) = 1
              f
C                 sgn(LB2 ) = 1
      Substituting the above values into equation (4.83) gives
C
          GJ(3,3)= 1.0
          GJ(3,4)= -1.0
          GJ(4*NT+3,4*NT+3)= 1.0
          GJ(4*NT+3,4*NT+4)= 1.0
C
C     GLOBAL JACOBIAN IS SOLVED
C
      M1=1
      N2=4*NT+5
      IA=MAXRG
      IDGT=0
      CALL LEQT1F(GJ,M1,N2,IA,RSVG,IDGT,WK,IER)
C
```

```
C     UPDATE NODAL VARIABLES
C
      DO 20 I3=1,NT+1
        R(I3)= R(I3)-RSVG(4*(I3-1)+1)
        THETA(I3)= THETA(I3)-RSVG(4*(I3-1)+2)
        LB1(I3)= LB1(I3)-RSVG(4*(I3-1)+3)
        LB2(I3)= LB2(I3)-RSVG(4*(I3-1)+4)
20    CONTINUE
        DT=DT-RSVG(4*NT+5)
C
C     CONVERGENCE CRITERION IS COMPUTED.   Refer equation(4.53)
C
      DO 30 I=1,NT+1
          CONVN= CONVN+(RSVG(4*(I-1)+1)**2)
          CONVD= CONVD+(R(I)**2)
C
          CONVN= CONVN+(RSVG(4*(I-1)+2)**2)

          CONVD= CONVD+(THETA(I)**2)
C
          CONVN= CONVN+(RSVG(4*(I-1)+3)**2)
          CONVD= CONVD+(LB1(I)**2)
C
          CONVN= CONVN+(RSVG(4*(I-1)+4)**2)
          CONVD= CONVD+(LB2(I)**2)
C
```

```
30    CONTINUE
C
          CONVN= CONVN+(RSVG(4*NT+5)**2)
          CONVD= CONVD+(DT**2)
C
          CONV= CONVN/CONVD
          CONV= DSQRT(CONV)

      RETURN
      END
C     ******************************************************************
```

A SOLUTION TECHNIQUE FOR THE
MINIMUM-TIME CONTROL PROBLEM
OF AN R-THETA MANIPULATOR

by

ANUP SHETTY

B.S.,Wichita State University, 1985

AN ABSTRACT OF A MASTER'S THESIS

submitted in partial fulfillment of

the requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1987

ABSTRACT

This paper investigates the minimum time control of a two degree of freedom manipulator subject to control magnitude constraints. Two methods of solution, a continuous time method and a discrete time method are applied, and their relative merits are examined.

The mathematical model for the r-θ manipulator, a two degree of freedom manipulator operating in the horizontal x-y plane is developed.

The necessary conditions for the minimum time control of the manipulator are presented. Two formulations, the first and the second order formulations, are used.

The finite element method is developed for the discrete time simulation of the time optimal control of the manipulator. A combination of a grid search technique and Newton-Raphson iteration on the finite element equations is used to obtain the minimum time with the state and control histories . The discrete time solution is compared to the continuous time solution. The results of the computer simulations are presented, as well as recommendations for further study.